On the closures of finite permutation groups

Andrey Vasil'ev

Novosibirsk State University and Sobolev Institute of Mathematics

Shijiazhuang, China
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Setup

Ω is a finite set, $G \leq \mathrm{Sym}(\Omega)$, and m is a positive integer

G acts on Ω^m: $(\alpha_1, \ldots, \alpha_m)^g = (\alpha_1^g, \ldots, \alpha_m^g)$.

$\mathrm{Orb}_m(G)$ is the set of orbits of this action (m-orbits).

G and H from $\mathrm{Sym}(\Omega)$ are *m-equivalent* if $\mathrm{Orb}_m(G) = \mathrm{Orb}_m(H)$.

G and H are m-equivalent $\Rightarrow \langle G, H\rangle$ is m-equivalent to them.

H. Wielandt (1969): The *m-closure* $G^{(m)}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$, m-equivalent to G.

m-Closure problem

Given a finite permutation group G, find $G^{(m)}$.

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$\mathrm{Inv}_m(G) = (\Omega, \mathrm{Orb}_m(G))$ is a discrete structure on Ω consisting of (colored) m-ary relations and $G^{(m)}$ is its full automorphism group:

$$G^{(m)} = \{g \in \mathrm{Sym}(\Omega) : \Delta^g = \Delta, \Delta \in \mathrm{Orb}_m(G)\} = \mathrm{Aut}(\mathrm{Inv}_m(G))$$

Motivation

$\Gamma = (\Omega, E)$ is a graph
Ω is the vertex set, $E \subseteq \Omega^2$ is the edge (arc) set

For graphs $\Gamma = (\Omega, E)$ and $\Gamma' = (\Omega', E')$,
$\mathrm{Iso}(\Gamma, \Gamma') = \{f : \Omega \to \Omega' \text{ a bijection } \mid E^f = E'\}$.

Graph Iso

Given two graphs Γ and Γ', test whether $\mathrm{Iso}(\Gamma, \Gamma') = \varnothing$.

Babai (2015): There is $c > 0$ such that for graphs Γ and Γ' of size n the set $\mathrm{Iso}(\Gamma, \Gamma')$ can be found in a quasipolynomial time $n^{O(\log^2 n)}$.

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If $f \in \mathrm{Iso}(\Gamma, \Gamma') \neq \varnothing$, then $\mathrm{Iso}(\Gamma, \Gamma') = \mathrm{Aut}(\Gamma)f$.

Graph Iso is polynomial-time equivalent to

Graph Aut

Given graph Γ, find $\mathrm{Aut}(\Gamma)$.

Graph isomorphism problem and m-dim WL-algorithm

$c : \Omega^m \to \mathbb{N}$ is a coloring (m-coloring) of Ω^m for some $m \geq 1$.

Initial coloring c of a graph $\Gamma = (\Omega, E)$ with $|\Omega| = n$: for $\overline{\alpha}, \overline{\beta} \in \Omega^m$, $c(\overline{\alpha}) = c(\overline{\beta})$ if the map $\alpha_i \mapsto \beta_i$ induces $\Gamma|_{\{\overline{\alpha}\}} \simeq \Gamma|_{\{\overline{\beta}\}}$.

m-dim WL-algorithm

1. For all $\overline{\alpha} \in \Omega^m$, find a multiset $s(\overline{\alpha}) = \{\{s_\gamma(\overline{\alpha}) : \gamma \in \Omega\}\}$, where $s_\gamma(\overline{\alpha}) = (c(\gamma, \alpha_2, \ldots, \alpha_m), \ldots, c(\alpha_1, \ldots, \alpha_{m-1}, \gamma))$.
2. Define c': $c'(\overline{\alpha}) < c'(\overline{\beta}) \Leftrightarrow c(\overline{\alpha}) < c(\overline{\beta})$ or $s(\overline{\alpha}) \prec s(\overline{\beta})$.
3. Go to Step 1 if $|\mathrm{Im}(c)| \neq |\mathrm{Im}(c')|$; otherwise output c.

The complexity: $O(m^2 n^{m+1} \log n)$ (Immerman–Lander, 1990).

WL_m-closure of graph Γ is $\mathrm{WL}_m(\Gamma) = (\Omega, R)$, where R is a partition of Ω^m into the color classes obtained by m-dim WL-algorithm.

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$$\operatorname{Iso}(\Gamma, \Gamma') = \operatorname{Iso}(\mathrm{WL}_m(\Gamma), \mathrm{WL}_m(\Gamma')).$$

It suffices to deal with WL-stable discrete structures.

WL_m-closures and m-closures

Ω is a set of size n, $G \leq \mathrm{Sym}(\Omega)$, and m is a positive integer

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$\mathrm{Orb}_m(G)$ is the set of orbits of this action (m-orbits).

$\mathrm{Inv}_m(G) = (\Omega, \mathrm{Orb}_m(G))$ is WL_m-stable:

$\mathrm{WL}_m(\mathrm{Inv}_m(G)) = \mathrm{Inv}_m(G)$.

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m-Closure problem

Given a finite permutation group G, find $G^{(m)}$.

Input: generators of G (m is not a part of input)

Output: generators of $G^{(m)}$

Computational complexity is a function on $n = \deg(G) = |\Omega|$.

Elementary properties of m-closures

Let $G, H \leq \mathrm{Sym}(\Omega)$ and m a positive integer.

Taking m-closure is a closure operator:
$G \leq G^{(m)}$, $G^{(m)} = (G^{(m)})^{(m)}$, and $G \leq H \Longrightarrow G^{(m)} \leq H^{(m)}$.

The group G is m-closed if $G^{(m)} = G$.

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If $G \leq \mathrm{Sym}(\Omega)$ is m-transitive, then $G^{(m)} = \mathrm{Sym}(\Omega)$. In particular,

$$\Omega = \underbrace{\Delta_1 \cup \ldots \cup \Delta_s}_{1-\text{orbits}} \Longrightarrow G^{(1)} = \mathrm{Sym}(\Delta_1) \times \ldots \times \mathrm{Sym}(\Delta_s).$$

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Example: $\Omega = \mathbb{F}_q$, $G = \mathrm{AGL}_1(q) = \{x \mapsto ax + b : a \in \mathbb{F}_q^{\times}, b \in \mathbb{F}_q\}$
$\Rightarrow G^{(1)} = G^{(2)} = \mathrm{Sym}(\Omega)$, but $G^{(3)} = G$.

Known computational results

The 2-closure of a permutation group G of degree n can be found in time polynomial in n provided

1. G is nilpotent (Ponomarenko, 1994)
2. G is of odd order (Evdokimov–Ponomarenko, 2001)
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Given $m \geq 3$, the m-closure of a solvable permutation group of degree n can be found in time $n^{O(m)}$ (Ponomarenko–V., 2023).

Properties of m-closures of solvable groups

Let $G, H \leq \mathrm{Sym}(\Omega)$ and m a positive integer.

Suppose $m \geq 2$. Then

1. G is abelian $\Rightarrow$ $G^{(m)}$ is abelian
2. G is of odd order $\Rightarrow$ $G^{(m)}$ is of odd order
3. G is a p-group $\Rightarrow$ $G^{(m)}$ is a p-group (Wielandt, 1969)
4. G is nilpotent $\Rightarrow$ $G^{(m)}$ is nilpotent (2020).

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O'Brien–Ponomarenko–V.–Vdovin (2021): If $m \geq 3$ and G is solvable, then $G^{(m)}$ is solvable.

Babai–Luks algorithm

Luks (1982): Graph Iso can be solved in time n^c, where c depends only on $\deg(\Gamma)$.

Babai–Luks (1983): Let $H \leq \mathrm{Sym}(\Omega)$ be such that $|H^\Delta| \leq n^d$ for every its primitive section Δ, and let $\Gamma = (\Omega, E)$ be a graph. Then $\mathrm{Aut}(\Gamma) \cap H$ can be found in time n^c, where c depends on d.

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Corollary: Let $H \leq \mathrm{Sym}(\Omega)$ be such that $|H^\Delta| \leq n^d$ for every its primitive section Δ, $G \leq \mathrm{Sym}(\Omega)$, and $m \in \mathbb{N}$. Then $G^{(m)} \cap H$ can be found in time n^c, where c depends on m and d.

$G^{(m)} \cap H$ is called the relative m-closure of G with respect to H.

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Pálfy (1982): If G is a primitive solvable group, then $|G| \leq n^4$.

Inside the proof for solvable groups

A class $\mathfrak{X}$ of (abstract) groups is said to be complete if it is closed with respect to taking subgroups, quotients, and extensions.

$\mathfrak{X}_n$ is the class of permutation groups from $\mathfrak{X}$ of degree at most n.

Main Lemma (Ponomarenko-V.,2023): Let $m, n \in \mathbb{N}$, $m \geq 3$, and $\mathfrak{X}$ a complete class of groups closed with respect to taking m-closures. The m-closure of a permutation group in $\mathfrak{X}_n$ can be found in time $n^{O(m)}$ by accessing two oracles:

1. for finding the m-closure of each primitive basic group in $\mathfrak{X}_n$
2. for finding the relative m-closure of every group in $\mathfrak{X}_n$ with respect to any group in $\mathfrak{X}_n$.

A permutation group is basic if it cannot be embedded in a wreath product in the product action.

If $\mathfrak{X}$ is the class of solvable groups, then $\mathfrak{X}$ is complete.

By O'Brien–Ponomarenko–V.–Vdovin theorem, it is closed with respect taking m-closures for $m \geq 3$.

If $G \in \mathfrak{X}_n$ and $m \geq 3$, then $G^{(m)}$ can be found in time $n^{O(m)}$ in view of Main Lemma, since

1. If G is primitive basic, then either G is m-closed and $G^{(m)} = G$, or $n \leq d$ for some constant $d \Rightarrow G^{(m)}$ can be easily found.
2. Relative m-closure of every group in $\mathfrak{X}_n$ with respect to any group in $\mathfrak{X}_n$ can be found via the Babai–Luks algorithm because of Pálfy's theorem.

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What is the next step?

Alt(d)-free groups

For $d \geq 5$, an (abstract) group is called Alt(d)-free, if it does not contain section isomorphic to the alternating group of degree d.

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The class $\mathfrak{X}$ of Alt(d)-free groups is complete.

If $d \geq 25$, then the list of simple groups in $\mathfrak{X}$ includes

- the groups of order p for all primes p
- all sporadic groups
- all exceptional groups of Lie type
- all classical groups of dimension less than $d - 2$.
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Babai–Cameron–Pálfy (1982): If G is a primitive Alt(d)-free group, then $|G| \leq n^c$, where c depends only on d.

Main Results

Ponomarenko–Skresanov–V. (2024): Let $\mathfrak{X}$ be a complete class including all Alt(25)-free groups. Then the m-closure of every permutation group from $\mathfrak{X}$ belongs to $\mathfrak{X}$ for each $m \geq 4$.

Corollary: If G is an Alt(d)-free group with $d \geq 25$, then $G^{(m)}$ is Alt(d)-free group for $m \geq 4$.

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Corollary: If G is an Alt(d)-free group with $d \geq 25$, then $G^{(m)}$ is Alt(d)-free group for $m \geq 4$.

$m = 4$ is the best possible, because $\mathrm{AGL}_k(2)^{(3)} = \mathrm{Sym}(2^k)$.

$d = 25$ is the best possible (for $m = 4$), because 4-closure of the Alt(9)-free Mathieu group M_{24} is Sym(24).

Inside the proof

Let $H = G^{(4)}$. It suffices to show that $H \in \mathfrak{X}$.
We may assume that G and H are a primitive basic group, because

Let $K \leq \mathrm{Sym}(\Gamma)$ and $L \leq \mathrm{Sym}(\Delta)$. Then

1. $K \times L$ acts on $\Gamma \sqcup \Delta \Rightarrow (K \times L)^{(m)} = K^{(m)} \times L^{(m)}$, $m \geq 1$ (folklore);
2. $K \wr L$ acts on $\bigsqcup_{\delta \in \Delta} \Gamma_\delta \Rightarrow (K \wr L)^{(m)} = K^{(m)} \wr L^{(m)}$, $m \geq 2$ (Kalužnin–Klin, 1976);
3. $K \wr L$ acts on $\Gamma^\Delta \Rightarrow (K \wr L)^{(m)} \leq K^{(m)} \wr L^{(m)}$, $m \geq 3$ (Ponomarenko–V., 2022).

In fact, $K \wr L$ acts on $\Gamma^\Delta \Rightarrow (K \wr L)^{(m)} = K^{(m)} \wr L^{[r]}$, $m \geq 2$, where $L^{[r]}$ is the largest permutation group on Δ having the same orbits on the set of ordered partitions into at most r parts as L, and $r = \min\{|\operatorname{Orb}_m(K)|, |\Delta|\}$.

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If G is almost simple, we are done by $(*)$.

If G is in a diagonal action, then $\mathrm{Soc}(G) = S^k$, where S is nonabelian simple and $G/S \leq \mathrm{Out}\, S \times L$, where $L \leq \mathrm{Sym}(k)$.
If $k \leq 4$ or $\mathrm{Alt} \leq L$, then $H \in \mathfrak{X}$, otherwise Fawcett (2013) $\Rightarrow$ the base size number of G is 2 (i.e. $G_{\alpha_1,\alpha_2} = 1$ for some $\alpha_1, \alpha_2 \in \Omega$), so $H = G$.

Thus, G and H are affine groups with the same socle.

G and H are affine $\Rightarrow n = p^d$, $G = V : G_0$ and $H = V : H_0$, where $G_0, H_0 \leq \mathrm{GL}(V) = \mathrm{GL}_d(p)$ are irreducible and primitive.

Lemma. $H_0 = G_0^{(3)} \cap \mathrm{GL}(V)$

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it reamins to deal with the following cases:

- G_0 and H_0 are tensor products,
- G_0 and H_0 are of symplectic type,
- G and H are quasisimple.

Tensor products

Suppose that V is a vector space over a finite field $\mathbb{F}$, $G = V : G_0$ is a primitive affine group and $\mathbb{F}^\times \leq G_0 \leq \Gamma\mathrm{L}(V)$. For $m \geq 4$,

1. if G_0 stabilizes a tensor decomposition $V = X \otimes Y$ over $\mathbb{F}$, $\dim X \neq \dim Y$, and the m-closures of $X : (G_0)_X$ and $Y : (G_0)_Y$ belong to $\mathfrak{X}$, then $G^{(m)} \in \mathfrak{X}$;
2. if G_0 preserves a tensor decomposition $V = \bigotimes_{i=1}^{k} X$ over $\mathbb{F}$, $k \geq 2$, $\pi : G_0 \to \mathrm{Sym}(k)$, and the m-closures of $X : (G_0)_X$ and $\pi(G_0)$ belong to $\mathfrak{X}$, then $G^{(m)} \in \mathfrak{X}$.

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Suppose that V is a vector space over a finite field $\mathbb{F}$, $G = V : G_0$ is a primitive affine group and $\mathbb{F}^\times \leq G_0 \leq \Gamma\mathrm{L}(V)$. For $m \geq 4$,

1. if G_0 stabilizes a tensor decomposition $V = X \otimes Y$ over $\mathbb{F}$, $\dim X \neq \dim Y$, and the m-closures of $X : (G_0)_X$ and $Y : (G_0)_Y$ belong to $\mathfrak{X}$, then $G^{(m)} \in \mathfrak{X}$;
2. if G_0 preserves a tensor decomposition $V = \bigotimes_{i=1}^k X$ over $\mathbb{F}$, $k \geq 2$, $\pi : G_0 \to \mathrm{Sym}(k)$, and the m-closures of $X : (G_0)_X$ and $\pi(G_0)$ belong to $\mathfrak{X}$, then $G^{(m)} \in \mathfrak{X}$.

In fact, for m-equivalent groups G and H, $\pi(G_0)^{[r]} = \pi(H_0)^{[r]}$, where $L^{[r]}$ is the largest subgroup of $\mathrm{Sym}(k)$ having the same orbits as L on the set of ordered partitions into at most r parts as L, and $r = \min\{|\operatorname{Orb}_m((G_0)_X/\mathbb{F}^\times)|, k\}$.

Following an idea from Liebeck–Shalev (2002), we obtain

If G_0 and H_0 are of symplectic type, then either G and H are Alt(25)-free, or G has the base of size 3 and $H = G$.

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Finally, we prove

If G_0 and H_0 are quasisimple, then either G and H are Alt(25)-free, or $\mathrm{Soc}(G_0/Z(G_0)) = \mathrm{Soc}(H_0/Z(H_0))$.

These statements allow to complete the proof of the main result.

Following an idea from Liebeck–Shalev (2002), we obtain

If G_0 and H_0 are of symplectic type, then either G and H are $\mathrm{Alt}(25)$-free, or G has the base of size 3 and $H = G$.

Finally, we prove

If G_0 and H_0 are quasisimple, then either G and H are $\mathrm{Alt}(25)$-free, or $\mathrm{Soc}(G_0/Z(G_0)) = \mathrm{Soc}(H_0/Z(H_0))$.

These statements allow to complete the proof of the main result.

In particular, we have

If G is an $\mathrm{Alt}(d)$-free group with $d \geq 25$, then $G^{(m)}$ is $\mathrm{Alt}(d)$-free group for $m \geq 4$.

Can one compute $G^{(m)}$ efficiently?

Ponomarenko–V. (work in progress): For $m \geq 4$ and $d \geq 25$, the m-closure of an $\mathrm{Alt}(d)$-free group can be found in time n^c, where c depends on m and d.

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Recall that if $m, n \in \mathbb{N}$, $m \geq 3$, and $\mathfrak{X}$ is a complete class of groups closed with respect to taking m-closures, then the m-closure of a permutation group in $\mathfrak{X}_n$ can be found in time $n^{O(m)}$ by accessing two oracles:

1. for finding the m-closure of each primitive basic group in $\mathfrak{X}_n$
2. for finding the relative m-closure of every group in $\mathfrak{X}_n$ with respect to any group in $\mathfrak{X}_n$.

From our results and Babai–Cameron–Pálfy (1982): If $m \geq 4$ and G is a primitive $\mathrm{Alt}(d)$-free group with $d \geq 25$, then $|G^{(m)}| \leq n^c$, where c depends only on d.

Thus, the relative m-closure of G can be found via the Babai–Luks algorithm in time n^c, where c depends on m and d.

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