On the closures of finite permutation groups

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Let m be a positive integer and let Ω be a finite set. The *m-closure* $G^{(m)}$ of a permutation group $G \leq \mathrm{Sym}(\Omega)$ is the largest permutation group on Ω having the same orbits as G in its induced action on the cartesian power Ω^m. The m-closure of a permutation group can be considered as the full automorphism group of the set of all m-ary relations invariant with respect to G. So one of the motivations of studying m-closures comes from a computational problem in which one needs to find efficiently the automorphism group $\mathrm{Aut}(\mathfrak{S})$ of a given set $\mathfrak{S}$ of relations (generally speaking of different arities). In the case when all the relations of $\mathfrak{S}$ are binary, this problem is equivalent to the famous Graph Isomorphism Problem and can be solved by the Babai algorithm [1] in quasipolynomial time in the size of $\mathfrak{S}$. It is currently unknown whether the automorphism group of a graph can be found in polynomial time.

The notion of m-closure were suggested by Helmut Wielandt in the framework of the method of invariant relations which he considered as the one of main tools for studying actions of a group on a set, see [2]. In view of [2, Theorem 4.3],

$$\mathrm{Sym}(\Omega) \geq G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(m)} = G^{(m+1)} = \cdots = G, \tag{1}$$

for some $m < \deg G$, so the closures of the group G can be considered as its successive approximations.

It is clear that $G^{(1)}$ hardly provides a nice approximations of G, because the 1-closure of any transitive group is the full symmetric group $\mathrm{Sym}(\Omega)$ (the same holds true for the m-closure of any m-transitive group for all m). However, for $m \geq 2$, if G is an abelian group (respectively, a p-group, a group of odd order), then $G^{(m)}$ is an abelian group (respectively, a p-group, a group of odd order), see [2]. Recently, it was proved [3] that a similar statement is true for solvable groups if $m \geq 3$ (the example of 2-transitive solvable groups shows that $m = 2$ cannot be taken here).

From the computational point of view, the m-closure problem consists in finding the m-closure of a permutation group given by its generating set. Polynomial-time algorithms for finding the m-closure were constructed for the nilpotent groups [4], groups of odd order [5], and supersolvable groups [6].

Quite recently we solved the problem of finding m-closure for solvable permutation groups provided $m \geq 3$ [7]. The proof was based on the above mentioned result [3]. One of the key ingredients of the algorithm depends on controlling the order of the m-closure of a primitive group lying in the class under consideration. It follows from the main result of [8] that there is a polynomial upper bound on the order of a primitive group if its nonabelian composition factors are restricted. Thus, a natural generalization of our result on solvable groups would be the transition from the class of solvable groups to the class of groups with restricted nonabelian composition factors. The following result obtained jointly with Ilia Ponomarenko and Saveliy Skresanov is a step in this direction. Recall that a group G is $\mathrm{Alt}(d)$*-free*, if G does not contain a section isomorphic to the alternating group of degree d.

Theorem. *If G is an $\mathrm{Alt}(d)$-free group for $d \geq 25$, then $G^{(m)}$ is $\mathrm{Alt}(d)$-free group for $m \geq 4$.*

Note that the constant 4 in the theorem is the best possible and, if $m = 4$, then the same holds true for the constant 25, as the following examples show:

(i) The affine group $G = \mathrm{AGL}_n(2)$ is 3-transitive in its natural action on a linear space of dimension $n \geq 2$ over the field of order 2. It follows that $G^{(3)} = \mathrm{Sym}(2^n)$, so the theorem does not hold for $m \leq 3$.

(ii) The Mathieu group $G = M_{24}$ is $\mathrm{Alt}(9)$-free and acts on 24 points 5-transitively, so $G^{(4)} = \mathrm{Sym}(24)$ is not $\mathrm{Alt}(24)$-free.

The work is supported by the Russian Science Foundation, https://rscf.ru/en/project/24-11-00127/

References

[1] L. Babai, Groups, Graphs, Algorithms: The Graph Isomorphism Problem, Proc. ICM 2018, Rio de Janeiro, Vol. 3, 3303–3320, see also L. Babai, Graph Isomorphism in Quasipolynomial Time (2016), arXiv:1512.03547.

[2] H. Wielandt, Permutation groups through invariant relations and invariant functions, The Ohio State University (1969).

[3] E. A. O'Brien, I. Ponomarenko, A. V. Vasil'ev, E. Vdovin, The 3-closure of a solvable permutation group is solvable, *J. Algebra* **607** (2022) 618–637.

[4] I. Ponomarenko, Graph isomorphism problem and 2-closed permutation groups, *Applicable Algebra in Engineering, Communication and Computing* **5** (1994) 9–22.

[5] S. Evdokimov, I. Ponomarenko, Two-closure of odd permutation group in polynomial time, *Discrete Math.* **235(1-3)** (2001) 221–232.

[6] I. Ponomarenko, A. V. Vasil'ev, Two-closure of supersolvable permutation group in polynomial time, *Comp. Complexity* **29** (2020) Paper no. 5.

[7] I. Ponomarenko, A. V. Vasil'ev, On computing the closures of solvable permutation groups, *Internat. J. Algebra Comput.* **34** (2024) 137–145.

[8] L. Babai, P. J. Cameron, P. Pálfy, On the orders of primitive groups with restricted nonabelian composition factors, *J. Algebra*, **79(1)** (1982) 61–168.

[9] I. Ponomarenko, S. V. Skresanov, A. V. Vasil'ev, Closures of permutation groups with restricted nonabelian composition factors (2024), arXiv:2406.03780.