Algebras of Jordan and Monster type

Sergey Shpectorov

University of Birmingham

Genesis of Finite Simple Groups:
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Monster and Majorana algebras

In 2009 Ivanov introduced a class of *Majorana algebras*. These were modelled after the properties of the Griess algebra for the Monster, the largest sporadic finite simple group.

Among the axioms was the following remarkable table:

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

This was the first use of a fusion law in the axiomatics of a class of non-associative algebras.

Algebras of Monster type

In 2015 Hall, Rehren, and S. published two related papers focussing on two classes of axial algebras. Algebras of Monster type (α, β) are characterised as primitive axial algebras for the following fusion law $\mathcal{M}(\alpha, \beta)$:

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Jordan type η

Let $\mathbb{F}$ be a field of characteristic not 2 and $\eta \in \mathbb{F}$, $\eta \neq 1, 0$. Algebras of Jordan type η are primitive axial algebras for the Jordan type fusion law $\mathcal{J}(\eta)$:

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

Jordan algebras

Jordan algebras are commutative algebras satisfying the *Jordan identity*

$$x(yx^2) = (xy)x^2.$$

They were introduced by German physicist *Ernst Pascual Jordan*.

If a is an idempotent in a Jordan algebra A then A has the *Peirce decomposition* with respect to a:

$$A = A_1(a) \oplus A_0(a) \oplus A_{\frac{1}{2}}(a),$$

and the multiplication of eigenvectors is exactly as in the fusion law $\mathcal{J}(\frac{1}{2})$. Hence Jordan algebras generated by primitive idempotents are examples of algebras of Jordan type $\eta = \frac{1}{2}$.

Matsuo algebras

Consider again a field $\mathbb{F}$ of characteristic not equal to 2, $\eta \in \mathbb{F}$, $\eta \neq 1, 0$, and an arbitrary group (G, D) of 3-transpositions. Take D as a basis of an algebra $A = M_\eta(G, D)$ over $\mathbb{F}$ and define multiplication as:

$$c \cdot d = \begin{cases} c, & \text{if } |cd| = 1 \text{ (hence } c = d) \\ 0, & \text{if } |cd| = 2 \text{ (hence } cd = dc) \\ \frac{\eta}{2}(c + d - e), & \text{if } |cd| = 3 \text{ (and hence } c^d = d^c =: e) \end{cases}$$

These are called *Matsuo algebras*. The basis vectors $d \in D$ are primitive idempotents and they satisfy the fusion law $\mathcal{J}(\eta)$.

So $M_\eta(G, D)$ is an algebra of Jordan type η.

Intersection

We can think of the Matsuo algebras as the generic examples and Jordan algebras as the exceptional case arising only for $\eta = \frac{1}{2}$. Is there an intersection between the two families?

Theorem (DMR2017+Y2018)

The only Matsuo algebras that are Jordan algebras are $M_{\frac{1}{2}}(S_n)$, $M_{\frac{1}{2}}(3^2 : 2)$, and, only when $\operatorname{char} \mathbb{F} = 3$, all $M_{\frac{1}{2}}(3^n : 2)$, $n \geq 3$.

If G is a simply-laced Weyl group then $M_{\frac{1}{2}}(G)$ has a factor that is a Jordan algebra.

Theorem (GMS2023)

The above are the only examples of Matsuo algebras whose non-zero factors are Jordan algebras.

Original Conjecture

Based on these examples, the following question was put forward:

Question

Is it true that every algebra of Jordan type is either a Matsuo algebra or a Jordan algebra?

Relation to groups

Let A be an algebra of Jordan type and let $a \in A$ be an axis. We set

- $A_+ = A_1(a) \oplus A_0(a)$; and
- $A_- = A_\eta(a)$.

Then $A = A_+ \oplus A_-$ and, furthermore, $A_+A_+ \subseteq A_+$, $A_+A_- \subseteq A_-$, and $A_-A_- \subseteq A_+$. That is, we have a C_2 grading on A for each axis a.
Consequently, $\tau_a : A \to A$, that acts as identity on A_+ and as minus identity on A_-, is an involution (automorphism of order 2) of A.

The *Miyamoto group* of A is $\mathrm{Miy}(A) = \langle \tau_a \mid a \in X \rangle \leq \mathrm{Aut}(A)$.

2-Generated case

Let $J_\eta(\phi)$, $\phi \in \mathbb{F}$, be the algebra over $\mathbb{F}$ with basis x, y, s:

	x	y	s
x	y	$\frac{1}{2}x + \frac{1}{2}y + s$	πx
y	$\frac{1}{2}x + \frac{1}{2}y + s$	y	πy
s	πx	πy	πs

where $\pi = \phi - \phi\eta - \eta$.

Theorem (HRS2015b)

Let $A = \langle\langle a, b\rangle\rangle$ be an algebra of Jordan type η, where a and b are primitive axes. Let ϕa be the projection of b onto $A_1(a) = \mathbb{F}a$. Then there is a unique surjective homomorphism from $J(\phi)$ onto A sending x and y onto a and b, respectively.

Not all combinations of η and ϕ are possible: either $\eta = \frac{1}{2}$ or $\phi = \frac{\eta}{2}$.

Closed sets of axes

We can close the set X of axes by taking $\bar{X} = X^{\mathrm{Miy}(A)}$. In this way $\mathrm{Miy}(A)$ can be made to act faithfully on the set of axes. An *axet* is a closed set of axes $X = \bar{X}$ together with the action of $\mathrm{Miy}(A)$ on it and the τ-map $\tau : X \to \mathrm{Miy}(A)$.

Theorem (HSS2018a)

If A if an algebra of Jordan type then A is 1*-closed; i.e., axes from the axet span A.*

In other words, every element of A is a linear combination of axes.

Case $\eta \neq \frac{1}{2}$

Let A be of Jordan type $\eta \neq \frac{1}{2}$ and $a, b \in A$ be primitive axes.

Then either $ab = 0$ (algebra $2B \cong \mathbb{F} \oplus \mathbb{F} \cong M_\eta(2^2)$) or $\phi = \frac{\eta}{2}$ (algebra $3C(\eta) \cong M_\eta(S_3)$).

Theorem (HRS2015b+HSS2018a)

If $\eta \neq \frac{1}{2}$ then every algebra of Jordan type η is isomorphic to (a factor of) a Matsuo algebra.

Frobenius form

A non-zero (symmetric) bilinear form $(\cdot, \cdot)$ on an algebra A is called a *Frobenius form* if it associates with the algebra product:

$$(u, vw) = (uv, w)$$

for all $u, v, w \in A$.

Theorem (HSS2018b)

Every algebra A of Jordan type admits a unique Frobenius form such that $(a, a) = 1$ for each primitive axis in A.

Three generators

Theorem (GS2020)

For $\eta = \frac{1}{2}$, there exists a 9-dimensional Jordan algebra $J(\alpha, \beta, \gamma, \psi) = \langle\langle x, y, z \rangle\rangle$ such that every algebra $A = \langle\langle a, b, c \rangle\rangle$ of Jordan type half (where a, b, and c are primitive axes) is the image of $J(\alpha, \beta, \gamma, \psi)$ under a homomorphism sending x, y, and z to a, b, and c, respectively. Here $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (a, c)$, and $\psi = (a, bc) = (b, ac) = (c, ab)$.

All structure constants of $J(\alpha, \beta, \gamma, \psi)$ are explicit polynomials in the four parameters.

$J(\alpha, \beta, \gamma, \psi)$ is simple if and only if

$$(\alpha + \beta + \gamma - 2\psi - 1)(\alpha\beta\gamma - \psi^2) \neq 0,$$

and then it is a form the 3×3 matrix Jordan algebra.

Four generators

The case of four generators is especially important, because this is where Matsuo and Jordan algebras diverge.

Theorem (DMRS2023)

A 4-generated algebra of Jordan type has dimension at most 81.

They construct an explicit spanning set, but do not exhibit an universal object, as in the case of two and three generators.

Question

Determine the universal 4-generated algebra of Jordan type half.

The bound of 81 is exact, because $M_{\frac{1}{2}}([3^8]:2)$ has dimension 81.

Universal baric algebra

An algebra A over $\mathbb{F}$ is called *baric* when it has a surjective *weight homomorphism onto* $\mathbb{F}$*:*

$$wt : A \to \mathbb{F}.$$

The kernel of wt is known as the baric radical of A.

In the axial case, we additionally require that none of the generating axes is in the baric radical. Then $wt(a) = 1$ for all axes a.

Theorem (SS2024?)

For any $k \geq 0$, there exists a universal k-generated baric algebra of Jordan type $\frac{1}{2}$. It is a Jordan for all k.

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In each characteristic other than 3, the universal 4-generated baric algebra of Jordan type half has dimension 54.

Solid 2-generated subalgebras

Definition

For axes $a, b \in A$, the subalgebra $U = \langle\langle a, b \rangle\rangle$ is called *solid* if and only if every primitive idempotent of U is a primitive axis of A, even over arbitrary extensions of $\mathbb{F}$.

Note that this definition is independent of the choice of the two specific generating axes of U.

Theorem (GSS2024,D2024)

Suppose that A is an algebra of Jordan type $\frac{1}{2}$. For axes $a, b \in A$, the subalgebra $U := \langle\langle a, b \rangle\rangle$ is non-solid only if $(a, b) = \frac{1}{4}$ and U contains exactly three axes: a, b, and $c = a^{\tau_b} = b^{\tau_a}$.

We will call such "non-solid" triples $\{a, b, c\}$ *Fischer triples*.

Derivations

For axes a, b of A, define $D_{ab} = [\mathrm{ad}_a, \mathrm{ad}_b] = \mathrm{ad}_a\mathrm{ad}_b - \mathrm{ad}_b\mathrm{ad}_a$.

Theorem (D2024)

The subalgebra $U = \langle\langle a, b\rangle\rangle$ is solid if and only if D_{ab} is a derivation of the entire algebra A.

This has an important corollary.

Theorem (D2024)

An algebra A of Jordan type $\frac{1}{2}$ is Jordan if and only if it has a spanning set X of generating axes such that $\langle\langle a, b\rangle\rangle$ is solid for all $a, b \in X$.

Geometry

Consider the point-line geometry, where points are the primitive axes from $X = \bar{X}$ and lines correspond to the 2-generated subalgebras $\langle\langle a, b\rangle\rangle$, $a, b \in X$.

Then the lines can be classified into several types:

- baric $J(1)$ and flat baric $\overline{J(1)}$;
- double (flat) baric $J(0)$ and orthogonal pairs $\overline{J(0)} \cong 2B$;
- toric $J(\alpha)$, $\alpha \neq 0, 1$; and
- Fischer triples $J(\frac{1}{4}) \cong 3C(\frac{1}{4})$.

We distinguish the *continuous* lines (baric and toric) and the *discontinuous* ones (double baric, orthogonal pairs and Fischer triples).

Components

Consider the collinearity graph on $X = \bar{X}$ with respect to continuous lines only. Let this have connected components $X_1, X_2, \ldots, X_k$. Let $C_i = \langle\langle X_i \rangle\rangle$. We call the C_i the *components* of A.

Theorem (DS2024?)

Each component C_i is a Jordan algebra.

Even more interesting is the following.

Theorem (DS2024?)

The set of components carries the structure of a Fischer space.

New paradigm

These results completely alter the paradigm:

- Matsuo algebras arise when components are 1-dimensional;
- Jordan algebras arise when there is only one component.

It looks like there could be middle case: (connected) Fischer spaces with more than one point and components of dimension greater that 1.

So the focus has now shifted:

Question

Which Jordan components can arise for which (connected) Fischer spaces?

Examples

Example (GS2020+D2022)

Consider $A = M_{\frac{1}{2}}(3^{n-1} : S_n)$. Here all *vertical* lines are solid and all *horizontal* lines are orthogonal pairs and Fischer triples. So we have a middle case: non-trivial components for a non-trivial Fischer space.

Desmet (2024?) explicitly constructed examples where the Fischer space is that of a simply connected Weyl group and components are baric lines. For diagrams D_n and E_n, these examples are neither Matsuo, not Jordan.

However, these examples are non-simple: modulo the radical, the components turn into 1-dimensional algebras $\mathbb{F}$ and the entire algebra becomes a Matsuo algebra. Hence the original conjecture may still be true for simple algebras of Jordan type.

A triality example?

Every subspace of the Fischer space of components defines a subalgebra. Hence the interesting case arises when we take a Fischer line.

Theorem (SW2025?)

If the Fischer space of components is a line then the Miyamoto group of A is a group with triality.

This leads to the following

Question

Is there an algebra A of Jordan type half with the Miyamoto group $O_8^+(\mathbb{F}) : S_3$?

This algebra would likely be simple.

Thank you all for listening!