Projective embeddings of long-root geometries

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Sijiazhuang, 1-5 December 2024

Preliminaries on projective embeddings

$\Gamma = (P, \mathcal{L})$ point-line geometry, P the point-set and $\mathcal{L}$ the set of lines.

Assumption: Γ is connected and no two lines are incident with the same set of points (accordingly, we regard the lines of Γ as subsets of P).

A *projective embedding* of Γ (also *embedding* for short) is an **injective** mapping e from P to the point-set of a projective geometry $\mathrm{PG}(V)$ such that

$e(P)$ spans $\mathrm{PG}(V)$ and maps every line of Γ **onto** a line of $\mathrm{PG}(V)$.

If $\mathbb{K}$ is the underlying division ring of the vector space V, we say that e is *defined over* $\mathbb{K}$.

Covers and projections of embeddings

For $i = 1, 2$ let $e_i : \Gamma \to \mathrm{PG}(V_i)$ be an embedding of Γ. We say that e_1 *covers* e_2 (also e_2 is a *projection* of e_1) if e_1 and e_2 are defined over the same division ring and there exists a semi-linear mapping $f : V_1 \to V_2$ such that

$$e_2 \;=\; \mathrm{pg}(f) \circ e_1$$

where $\circ$ stands for composition and $\mathrm{pg}(f) : \mathrm{PG}(V_1) \to \mathrm{PG}(V_2)$ is the morphism of projective geometries associated to f.
If e_1 covers e_2 we write $e_1 \geq e_2$. If f is bijective then we say that e_1 and e_2 are *isomorphic* and we write $e_1 \cong e_2$.

Remark

The semi-linear mapping f, if it exists, is uniquely determined by e_1 and e_2 up to scalars, namely the morphism $\mathrm{pg}(f)$ is unique. Accordingly, we have $e_1 \geq e_2 \geq e_1$ if and only if $e_1 \cong e_2$.

If $e_1 \geq e_2 \ncong e_1$ then we say that e_1 *properly covers* e_2.

Absolutely and relatively universal embeddings

An embedding $\widetilde{e} : \Gamma \to \mathrm{PG}(\widetilde{V})$ is *absolutely universal* (*absolute* for short) if it covers all embeddings of Γ.

Clearly, the absolute embedding, if it exists, is unique up to isomorphisms. It is also clear that a geometry Γ admits the absolute embedding only if all of its embeddings are defined over the same division ring.

An embedding $\widehat{e} : \Gamma \to \mathrm{PG}(\widehat{V})$ is *relatively universal* if it admits no proper cover. Equivalently, for every projection e of $\widehat{e}$, the embedding $\widehat{e}$ also covers all embeddings which cover e:

$$(e \leq \widehat{e}) \wedge (e \leq e') \;\Rightarrow\; e' \leq \widehat{e}.$$

If e is an embedding of Γ and $\widehat{e} \geq e$ is relatively universal then, up to isomorphism, $\widehat{e}$ is the unique relatively universal embedding that covers e. We call it the *universal cover* of e.

Existence of universal embeddings

Theorem (Ronan 1987)

Every embedding admits the universal cover.

Clearly, absolute embeddings are relatively universal. An absolute embedding of Γ is indeed the universal cover of all embeddings of Γ. So, Γ admits the absolute embedding if and only if all of its embeddings admit the same universal cover.

Far reaching sufficient conditions for the existence of the absolute embedding have been found by A. Kasikova and E. Shult (2001). I am not going to recall those conditions here. I only mention that when Γ is an embeddable parapolar space they boil down to the following:

(*) every circuit of the collinearity graph of Γ splits in triangles and proper quadrangles, each of the latter being contained in a convex polar subspace (symp) of Γ other than a grid.

In many of the parapolar spaces investigated in the literature every circuit indeed splits in triangles and quadrangles. For these spaces condition $(*)$ can be freely rephrased as follows:

- the space Γ contains 'sufficiently many' symps which are not grids.

Of course, this is a very vague formulation but in practise it is always clear if enough symps other than grids exist so that to fulfill condition $(*)$.

Lie geometries

Given a building Δ of spherical type, let $J \neq \emptyset$ be a nonempty subset of the set of types of Δ.

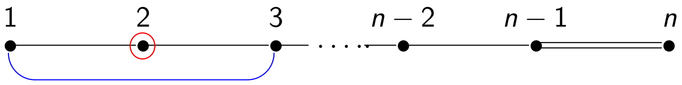
We can define a point-line geometry Δ_J as follows:

- points of Δ_J: the J-flags of Δ;
- lines: the flags of type $j^{\sim} \cup (J \setminus \{j\})$, for $j \in J$ and $j^{\sim}$ the adjacency of j in the Coxeter diagram of Δ;
- incidence: incidence between flags as in Δ.

Geometries constructed in this way are called *Lie geometries*.

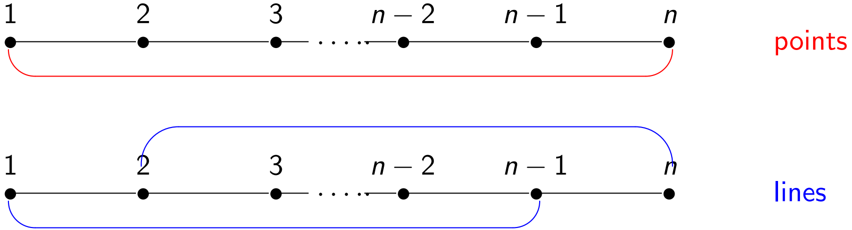
Example

With Δ the building of a polar space of rank $n \geq 3$, type-set $\{1, 2, ..., n\}$ and $J = \{2\}$, the points of $\Delta_J = \Delta_2$ are the lines of Δ and the lines of Δ_2 are the point-plane flags of Δ.

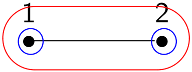


More examples

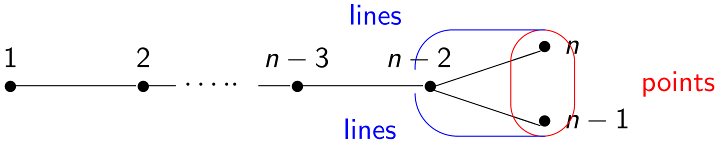
With Δ an n-dimensional projective geometry and $J = \{1, n\}$, the points of $\Delta_J = \Delta_{\{1,n\}}$ are the point-hyperplane flags of Δ and the lines of $\Delta_{\{1,n\}}$ are the line-hyperplanes flags and the point-subhyperplane flags of Δ.



In particular, when $n = 2$ the points and the lines of Δ_J are respectively the chambers and the elements of Δ.



Δ of type D_n and $J = \{n-1, n\}$.



Here the points of Δ_J can be identified with the sub-generators of the polar space associated with Δ while the lines of Δ_J consist of a generator and a sub-sub-generator contained in it.

Lie geometries; notation

The situation most frequently considered in the literature is the following:

Δ is associated with a split Chevalley group and $|J| = 1$. When Δ is of type A_n or D_n the cases $J = \{1, n\}$ (for Δ of type A_n) and $J = \{n-1, n\}$ (when Δ is of type D_n) are also of interest.

In the sequel the symbols used for diagrams are as in Dynkin notation.

Given a Dynkin diagram X_n of rank n and a field $\mathbb{F}$ we denote by $X_n(\mathbb{F})$ the building of (Dynkin) type X_n defined over $\mathbb{F}$. Given a set J of types of $X_n(\mathbb{F})$, $X_{n,J}(\mathbb{F})$ is the geometry Δ_J with $\Delta = X_n(\mathbb{F})$.

For instance:

$\Delta_J = B_{n,k}(\mathbb{F})$ with $J = \{k\}$ and $\Delta = B_n(\mathbb{F})$ (the elements of which are the singular subspaces of the polar space associated with the orthogonal group $O(2n+1, \mathbb{F})$);

$\Delta_J = A_{n,\{1,n\}}(\mathbb{F})$ with $J = \{1, n\}$ and $\Delta = A_n(\mathbb{F})$ (the elements of which are the proper nonempty subspaces of $\mathrm{PG}(n, \mathbb{F})$.

In particular, $A_{n,\{1,n\}}(\mathbb{F})$, $C_{n,1}(\mathbb{F})$, $B_{n,2}(\mathbb{F})$, $D_{n,2}(\mathbb{F})$, $F_{4,1}(\mathbb{F})$, $E_{6,2}(\mathbb{F})$, $E_{7,1}(\mathbb{F})$, $E_{8,8}(\mathbb{F})$ and $G_{2,2}(\mathbb{F})$ are the so-called *long-root geometries* for $\mathrm{SL}(n+1, \mathbb{F})$, $\mathrm{Sp}(2n, \mathbb{F})$, $\mathrm{O}(2n+1, \mathbb{F})$, $\mathrm{O}^+(2n, \mathbb{F})$ and the Chevalley groups of type F_4, E_6, E_7, E_8 and G_2 respectively.

A naive belief.

All embeddable Lie geometries admit the absolute embedding, which is the one hosted by the appropriate Weyl module, except possibly when $\mathbb{F}$ is very small (e.g. $|\mathbb{F}| = 2$).

FALSE! The above is true for many Lie geometries but not for all of them.

As regards the existence of the absolute embedding, when $|\mathrm{Aut}(\mathbb{F})| > 1$ the long-root geometry $A_{n,\{1,n\}}(\mathbb{F})$ is a counterexample to the above claim. Most likely, $D_{n,\{n-1,n\}}(\mathbb{F})$ with $|\mathrm{Aut}(\mathbb{F})| > 1$ is also a couterexample.

Remark

Turning back to $(*)$ of my fifth slide, both $A_{n,\{1,n\}}(\mathbb{F})$ $(n > 2)$ and $D_{n,\{n-1,n\}}(\mathbb{F})$ are parapolar spaces but in each of them all symps are grids. Regarding $A_{2,\{1,2\}}(\mathbb{F})$, this geometry is a non-thick generalized hexagon; it admits no quadrangles.

Moreover, when Δ_J is a long-root geometry, it can happen that the embedding hosted by the adjoint module (which is here the appropriate one) is not relatively universal. Hence this embedding cannot be absolutely universal even if Δ_J admits the absolute embedding.

Remark

In view of Kasikova-Shult's conditions, all long-root geometries but possibly those of type $A_{n,\{1,n\}}$ admit the absolute embedding.

Thus, eventually, I've come to the core of my talk.

Theorem (Blok and P. , 2003)

When $\mathbb{F}$ is a prime field both $A_{n,\{1,n\}}(\mathbb{F})$ ($n > 2$) and $D_{n,\{n-1,n\}}(\mathbb{F})$ admit the absolute embedding.

The same holds true for $A_{2,\{1,2\}}(\mathbb{F})$ with $\mathbb{F}$ a prime field (J. Thas and H. Van Maldeghem 2000).

Let now $\mathbb{F}$ be arbitrary and let $M^0_{n+1}(\mathbb{F})$ be the adjoint module for the linear group $\mathrm{SL}(n+1,\mathbb{F})$. Namely $M^0_{n+1}(\mathbb{F})$ is the underlying vector space of the Lie algebra $\mathfrak{sl}(n+1,\mathbb{F})$. Recall that $M^0_{n+1}(\mathbb{F})$ consists of the traceless square matrices of order $n+1$ with entries in $\mathbb{F}$ and it is a hyperplane of the space $M_{n+1}(\mathbb{F})$ of all square matrices of order $n+1$ with entries in $\mathbb{F}$.
The group $\mathrm{SL}(n+1,\mathbb{F})$ acts on it by conjugation.

The natural embedding of $A_{n,\{1,n\}}(\mathbb{F})$

The geometry $A_{n,\{1,n\}}(\mathbb{F})$ admits an embedding e in $\mathrm{PG}(M^0_{n+1}(\mathbb{F}))$, defined as follows.

Given a point $\{p, H\}$ of $A_{n,\{1,n\}}(\mathbb{F})$, let $a \in V = V(n+1, \mathbb{F})$ be a representative vector of p and $\alpha \in V^*$ a linear functional representing H. Regarded the vectors of V as rows and those of V^* as columns, we can consider the matrix $\alpha \cdot a$ (row-times-column product). Note that $\mathrm{Tr}(\alpha \cdot a) = a \cdot \alpha = \alpha(a) = 0$ (because $p \in H$ by choice). The embedding e maps $\{p, H\}$ onto the point of $\mathrm{PG}(M^0_{n+1}(\mathbb{F}))$ represented by $\alpha \cdot a$:

$$e(\{p, H\}) \ = \ \langle \alpha \cdot a \rangle.$$

I call this embedding the *natural embedding* of $A_{n,\{1,n\}}(\mathbb{F})$.

If σ is a non-trivial automorphism of $\mathbb{F}$, we can also consider a *twisted* version of e_σ of e by setting

$$e_\sigma(\{p, H\}) \;=\; \langle \alpha \cdot a^\sigma \rangle.$$

(J. Thas and H. Van Maldeghem 2000.) The mapping e_σ embeds $A_{n,\{1,n\}}(\mathbb{F})$ in $\mathrm{PG}(M_{n+1}(\mathbb{F}))$.

Lemma (P., 2024)

The embeddings e and e_σ admit no common cover.

Therefore

Theorem

If $|\mathrm{Aut}(\mathbb{F})| > 1$ then $A_{n,\{1,n\}}(\mathbb{F})$ admits no absolute embedding.

Conjecture

The same holds true for $D_{n,\{n-1,n\}}(\mathbb{F})$: if $|\mathrm{Aut}(\mathbb{F})| > 1$ then $D_{n,\{n-1,n\}}(\mathbb{F})$ admits no absolute embedding.

Problem

What can we say when $\mathbb{F}$ is non-prime but admits no non-trivial automorphism? In particular:

- What about the case of $\mathbb{F} = \mathbb{R}$?
- Is the existence of non-trivial endomorphisms of $\mathbb{F}$ enough for $A_{n,\{1,n\}}(\mathbb{F})$ to admit no absolute embedding?

Next question

Is the natural embedding $e : A_{n,\{1,n\}}(\mathbb{F}) \to \mathrm{PG}(M^0_{n+1}(\mathbb{F}))$ relatively universal?
Again, the answer is: it depends on $\mathbb{F}$ (Smith and Völkein 1989 for $n = 2$, Cardinali, Guzzi and P. 2024 for the general case).
Explicitly,

Theorem

The embedding e is relatively universal if and only if $\mathbb{F}$ is either perfect of positive characteristic or an algebraic extension of the field of rational numbers.

Recall that a *derivation* of a field $\mathbb{F}$ is an additive mapping $d : \mathbb{F} \to \mathbb{F}$ such that

$$d(xy) \;=\; d(x)y + xd(y), \quad \forall x, y \in \mathbb{F}.$$

The derivations of $\mathbb{F}$ form an $\mathbb{F}$-vector space $\mathrm{Der}(\mathbb{F})$.

Let $K_{\mathrm{der}}(\mathbb{F})$ be the largest subfield of $\mathbb{F}$ such that all derivations of $\mathbb{F}$ induce the null map on it. When $\mathrm{char}(\mathbb{F}) = 0$ then $K_{\mathrm{der}}(\mathbb{F})$ is the algebraic closure (in $\mathbb{F}$) of the minimal subfield of $\mathbb{F}$. When $\mathrm{char}(\mathbb{F}) = p > 0$ then $K_{\mathrm{der}}(\mathbb{F}) = \mathbb{F}^p$.
So, $\mathrm{Der}(\mathbb{F}) = \{0\}$ if and only if $\mathbb{F}$ is either perfect of positive characteristic or algebraic over the field of rationals.

Let $\Omega \subseteq \mathbb{F} \setminus K_{\mathrm{der}}(\mathbb{F})$ be such that every mapping $\nu : \Omega \to \mathbb{F}$ extends to a unique derivation d_ν of $\mathbb{F}$. I call such a set a *derivation basis* of $\mathbb{F}$.

Remark

When $\mathrm{char}(\mathbb{F}) = 0$ the derivation bases of $\mathbb{F}$ are just the transcendence bases of $\mathbb{F}$ over its minimum subfield. When $\mathrm{char}(\mathbb{F}) = p > 0$ the derivation bases are the sets $X \subset \mathbb{F} \setminus \mathbb{F}^p$ such that $\mathbb{F}^p \cup X$ generates $\mathbb{F}$ as a field and are minimal with respect to this property.
In both cases, all derivations bases of $\mathbb{F}$ have the same cardinality.

Let Ω be a derivation basis of $\mathbb{F}$. Then $\mathrm{Der}(\mathbb{F}) \cong \mathbb{F}^{\Omega}$ (the space of $\mathbb{F}$-valued mappings over Ω).

For every $\omega \in \Omega$, let d_ω be the derivation which maps ω onto 1 and all of $\Omega \setminus \{\omega\}$ onto 0. Let $\mathrm{Der}_\Omega(\mathbb{F})$ be the subspace of $\mathrm{Der}(\mathbb{F})$ spanned by $\{d_\omega\}_{\omega\in\Omega}$.

So, $\mathrm{Der}_\Omega(\mathbb{F})$ consists of the derivations which are null on all but a finite number of elements of Ω and $\mathrm{Der}(\mathbb{F}) \cong (\mathrm{Der}_\Omega(\mathbb{F}))^*$.

Put $A := M^0_{n+1}(\mathbb{F})$ for short and $\widetilde{A} := \mathrm{Der}_\Omega(\mathbb{F}) \times A$.

Recall that $G = \mathrm{SL}(n+1, \mathbb{F})$ acts by conjugation on A:

$$a \in A \ \stackrel{g}{\longrightarrow}\ g^{-1}ag \in A$$

where $g \in G$ is regarded as a non singular matrix of order $n+1$.

An action of G on $\widetilde{A}$ can be defined as follows:

$$(d, a) \in \widetilde{A} \ \xrightarrow{g} \ \left(d + \sum_{\omega \in \Omega} \mathrm{Tr}(g \cdot d_\omega(g^{-1}) \cdot a) d_\omega, \ \ g^{-1} a g \right)$$

where for a matrix $x = (x_{i,j})_{i,j=1}^{n+1}$ we put $d_\omega(x) := (d_\omega(x_{i,j}))_{i,j=1}^{n+1}$.

Remark

Every element $t \in \mathbb{F}$ belongs to $\langle K_{\mathrm{der}}(\mathbb{F}) \cup \Omega_t \rangle$ for a finite subset Ω_t of Ω. So, $d_\omega(t) = 0$ for all but a finite number of choices of $\omega \in \Omega$. Accordingly, for every matrix x we have $d_\omega(x) = O$ (null matrix) for all but a finite number of choices of $\omega \in \Omega$.

We define an embedding $\widetilde{e}$ of $A_{n,\{1,n\}}(\mathbb{F})$ in $\mathrm{PG}(\widetilde{A})$ as follows: if $e(\{p, H\})$ is represented by the matrix $\alpha \cdot a \in A$, then $\widetilde{e}(\{p, H\})$ is represented by the following element of $\widetilde{A}$:

$$\left(\sum_{\omega \in \Omega} a \cdot d_\omega(\alpha) \cdot d_\omega, \ \ \alpha \cdot a \right).$$

Theorem (Cardinali, Giuzzi and P., 2024)

The embedding $\widetilde{e}$ defined as above is the universal cover of e.

Corollary

The natural embedding e is relatively universal if and only if $\mathrm{Der}(\mathbb{F}) = \{0\}$, namely if and only if $\mathbb{F}$ is either perfect of positive characteristic or an algebraic extension of the field of rational numbers.

When $\mathrm{char}(\mathbb{F}) \neq 2$ a similar result also holds for $B_{n,2}(\mathbb{F})$ and $D_{n,2}(\mathbb{F})$. I shall only discuss the B_n-case, but what I'll say for it can be repeated for D_n word by word.

Assume that $\mathrm{char}(\mathbb{F}) \neq 2$ and put $\Delta = B_n(\mathbb{F})$, $G = O(2n+1, \mathbb{F})$, A is the adjoint module for G, namely the underlying G-module of the Lie algebra $\mathfrak{o}(2n+1, \mathbb{F})$, and $V = V(2n+1, \mathbb{F})$.
Also, $\widetilde{A} := \mathrm{Der}_\Omega(\mathbb{F}) \times A$, just as in the case of $A_{n,\{1,n\}}(\mathbb{F})$.
Likewise in the $A_{n,\{1n\}}$-case, the group G acts by conjugation on A. Its action on $\widetilde{A}$ is defined just as in the $A_{n,\{1,n\}}$-case.

The natural embedding e of the long-root geometry Δ_2 in $\mathrm{PG}(A)$ can be described as follows.

Let $a, b \in V$ be such that $\langle a, b\rangle$ is a (singular) line of the polar space Δ_1 associated with G. Then e maps $\langle a, b\rangle$ onto the point of $\mathrm{PG}(A)$ represented by the matrix

$$J \cdot (a^T b - b^T a) \;=\; (aJ)^T b - (bJ)^T a$$

where the vectors of V are treated as rows, T stands for transposition and

$$J \;=\; \begin{pmatrix} O_n & I_n & \mathbf{0}^T \\ I_n & O_n & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$$

is the representative matrix of the symmetric bilinear form preserved by G. (Of course, O_n and I_n are the null square matrix and identity matrix of order n while $\mathbf{0}$ is the null vector of V.)

The universal cover $\widetilde{e}$ of e is the absolute embedding of Δ_2, since Δ_2 admits the absolute embedding (Kasikova and Shult 2001).

Theorem (P., September 2024)

The embedding $\widetilde{e}$ lives in the G-module $\widetilde{A} = \mathrm{Der}_\Omega(\mathbb{F}) \times A$ and maps the point $\langle a, b\rangle$ of Δ_2 onto the point of $\mathrm{PG}(\widetilde{A})$ represented by the following element of $\widetilde{A}$:

$$\left(\sum_{\omega\in\Omega}\left(b\cdot(d_\omega(a)J)^T - a\cdot(d_\omega(b)J)^T\right)\cdot d_\omega, \quad J(a^Tb - b^Ta)\right).$$

Problem

In the above I assume $\mathrm{char}(\mathbb{F}) \neq 2$. What can we say when $\mathrm{char}(\mathbb{F}) = 2$?

The symplectic case

A similar game can be played with $C_{n,1}(\mathbb{F})$ but in this case we must switch from projective embeddings to **veronesean** embeddings, where the lines of the geometry to be embedded are mapped onto **non-singular conics**.

Indeed the natural embedding of $C_{n,1}(\mathbb{F})$ in the (projective space of the) Lie algebra $\mathfrak{sp}(2n, \mathbb{F})$ is veronesean. Explicitly, it is the composition of the natural (projective) embedding of the polar space $C_{n,1}(\mathbb{F})$ in $\mathrm{PG}(2n-1, \mathbb{F})$ with the mapping from $\mathrm{PG}(2n-1, \mathbb{F})$ to the Veronese variety $\mathcal{V}_{2n-1}$.

Remark

Ronan's existence proof of the universal cover of a projective embedding can easily be rephrased for veronesean embeddings. I don't know if the conditions found by Kasikova and Shult for the existence of the absolute projective embedding can be rephrased in such a way that they also work for veronesean embeddings.

More problems

Problem

What about the long root geometries of exceptional type?
Their natural embeddings are projective and, as proved by Völklein (1989), when $\mathrm{Der}(\mathbb{F}) = \{0\}$ these embeddings are relatively universal (hence also absolutely universal, since these geometries admit the absolute embedding). Does the converse hold too?

Problem (a vague problem)

Why oddish stuations like those I have described in the last part of my lecture, where the natural embedding fails to be universal, only occur with long root geometries?
Is there any peculiar property of these geometries or their natural embeddings which can be pointed out as responsible for this?

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Thanks for your attention