Projective embeddings of long root geometries

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According to a widely held belief, nothing unexpected can occur with projective embeddings of Lie geometries, except possibly when they are defined over a very small field (e.g., over $\mathbb{F}_2$). In particular, it is supposed that all of them admit the absolutely universal embedding (from which all other projective embeddings of the given geometry can be obtained as projections) and this absolute embedding is the one hosted by the appropriate Weyl module. In my talk I will present two results which disprove this belief. (1) The long-root geometry $A_{n,\{1,n\}}(\mathbb{F})$ for $\mathrm{SL}(n+1,\mathbb{F})$ (where the points are the point-hyperplane flags of $\mathrm{PG}(n,\mathbb{F})$) admits the absolutely universal embedding only if the field $\mathbb{F}$ admits no non-trivial automorphism. (2) The embedding of $A_{n,\{1,n\}}(\mathbb{F})$ in (the projective geometry of the underlying vector space of) the lie Algebra $\mathfrak{sl}(n+1,\mathbb{F})$ is relatively universal, namely it is not a projection from a larger embedding, if and only if $\mathbb{F}$ admits no non-trivial derivation. When $\mathrm{char}(\mathbb{F}) \neq 2$ the same holds for the projective embeddings of the long root geometries $B_{n,2}(\mathbb{F})$ and $D_{n,2}(\mathbb{F})$ in the Lie algebras $\mathfrak{o}(2n+1,\mathbb{F})$ and $\mathfrak{o}(2n,\mathbb{F})$ respectively.

References

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