On finiteness conditions in groups

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Plan

The theory of finite groups is well developed and includes many in-depth results - just mention Genesis of Finite simple groups.

Some of the theorems proved first for finite groups can be transferred to broader classes of groups, imposing certain restrictions, weaker than the finiteness of the number of elements.

Such constraints are called **finiteness conditions**.

The talk is to review some recent research on finiteness conditions: and I will try to focus on the aspects related to finite simple groups and Majorana and axial algebras.

Definitions

A group G is said to be **locally finite**
if every finite subset of G generates a finite subgroup.

Considering subsets with just one element, note that such G is **periodic**
i.e. the order of any element in G must be finite.

The following definition is also used to bound element orders:
A group of **period** n is a group where the identity $x^n = 1$ holds.

Example: a group of period $2 \Rightarrow$ abelian $\Rightarrow$ locally finite.

Schematically these finiteness conditions are related as follows:

$$\text{finite} \subset \text{locally finite} \subset \text{periodic}$$

(strict inclusions).

Burnside problem

An interest to these finiteness conditions is very old.

(W.Burnside to R.Fricke, 1900): Can a group, generated by a finite number of operations, and such that the order of every one of its operations is finite and less than an assigned integer (n), consist of an infinite number of operations?

However, the problem is best known in the following terms:

(Burnside problem): Let n be a positive integer.
Is a group of period n locally finite?

Is a group of period n locally finite?

In general the Burnside problem is solved negatively:

odd $n \geq 665$ (P.S. Novikov, S.I. Adyan, 1978);
(so when you are 66 and 6 you already may not be locally finite
(but then you are odd) :)

even $n \geq 8000$ (I.G. Lysenok, 1996).

For $n = 3$ (W.Burnside 1902), $n = 4$ (I.N. Sanov 1940) and $n = 6$ (M.Hall 1958) the answer is positive.

The precise bound for n separating locally finite groups is still not known.

The Burnside problem can be thought of as the heavy obstacle on the way of generalizing finite group theory results to groups with some finiteness conditions.

Spectrum

Let's discuss some of the results in finite group theory and their analogs. In China and in Russia there is a lot of research on finite groups with a given spectrum. So I will start here. Recall,

A spectrum of a periodic group G is the set $\omega(G)$ of its element orders.

Of course, it is sufficient and convenient to list a smaller set: $\mu(G)$, which is the set of maximal by division elements of $\omega(G)$.

Example:, $\mu(A_7) = \{4, 5, 6, 7\}$ (alternating group).

However, early results on groups with a given spectrum were not on finite groups.

On groups with a given spectrum

(Bernhard Neumann, 1937) classified groups with $\omega(G) = \{1, 2, 3\}$, in particular, proved that they are locally finite.

Some further results were as follows:

(M.Newman, 1979) $\omega(G) = \{1, 2, 5\} \Rightarrow G$ is locally finite.

(N.D.Gupta, V.D.Mazurov, 1999) $\omega(G) \subset \{1, 2, 3, 4, 5\}$ (strict subset), then either G is locally finite, or it contains a nilpotent normal subgroup S s.t. G/S is 5-group.

(E.Jabara, 2004) If a group P of period 5 acts freely on an abelian $2, 3$-group, then $|P| = 5$.

Groups whose element orders are consequitive integers

G is called C_n-group, if $\omega(G) = \{1, 2, 3, 4, ..., n\}$.

Rolf Brandl and Shi Wujie in 1991 described finite C_n-groups. They proved that $n \leq 8$ and for $n = 1, ..., 8$ list all possibilities for G.

Classification of all (not necessarily finite) C_3 and C_4 groups follows from the results of B.Neumann and I.Sanov, respectively.
(V.D.Mazurov, 2000) C_5 groups are locally. finite.

In a series of work we classified all C_6 and C_7 groups.

C_6-groups

D.Lytkina, V.Mazurov, E.Jabara, A.M., 2014

If spectrum of G equals $\{1, 2, 3, 4, 5, 6\}$, then G is locally finite and one of the following holds:

- $N = O_5(G)$ is a nontrivial elementary abelian group, $G = NC$, where C is isomorphic to $SL_2(3)$ or $\langle x, y \mid x^3 = y^4 = 1, x^y = x^{-1} \rangle$, and C acts freely on N.
- $T = O_2(G)$ is a nontrivial elementary abelian group and G/T is isomorphic to A_5.
- G is isomorphic to S_5 or S_6.

C_7-groups and open questions

E.Jabara, A.M., 2021

C_7-group is isomorphic to A_7.

Another way to state this result is:

An alternating group A_7 is recognizable by spectrum in the class of <u>all</u> groups.

Two questions are still open in this direction:
Question 1: Is C_8-group locally finite?
Question 2: Does there exist C_n-group with $n > 8$?
(It should be not locally finite.)

Groups recognizable by spectrum

Many finite simple groups are known to be recognizable by spectrum in the class of finite groups.
V.D.Mazurov, A.Yu.Ol'shanskii, and A.I.Sozutov (2015) showed that $L_2(q)$ for some large q are not recognizable by spectrum in the class of all groups (but they are recognizable in the class of finite groups).
The following groups are known to be recognizable in the class of all groups:

- (A.H.Zhurtov, V.D.Mazurov, 1999) $PSL_2(2^n)$;
- (A.A.Kuznetsov, D.V.Lytkina, 2007) $PSL_2(7)$;
- (D.V.Lytkina, E.Jabara, A.M., 2014) Mathieu group M_{10} (maximal subgroup in finite simple group M_{11});
- (E.Jabara, A.M., 2015) $PSL_3(4) \simeq M_{11}$;
- (E.Jabara, A.M., 2021) A_7.

Restricting some 2-generated subgroups

Studying groups with a given period or spectrum, we put finiteness restrictions on subgroups generated by one element.
There are many remarkable results for **finite groups**, where the subgroups generated by the conjugacy class C are described, based on the constraints on the subgroups generated by two elements of C.
For example
Baer-Suzuki theorem:
the conjugacy class C in a finite group generates a nilpotent subgroup if any two elements of C generate a nilpotent subgroup.

Which of these results about finite groups can be extended to broader classes of groups? In other words, which of the conditions for subgroups generated by two elements from C are good finiteness conditions, for example, provide local finiteness of the group $\langle C \rangle$?

Baer-Suzuki theorem: formulations and consequences

Statement. *Let G be a finite group, $x \in G$, p be a prime.*
If $\forall g \in G$ a subgroup $\langle x, x^g \rangle$ is a p-group, then $\langle x^G \rangle$ is p-group.

The conclusion is equivalent to the fact that x falls into the p-radical $O_p(G)$, which is the maximal normal p-subgroup of G.

For $p = 2$, using the properties of dihedral groups, we get the classical

Corollary. *In a finite simple group an involution inverts some nontrivial element of odd order.*

Baer-Suzuki theorem for periodic groups: negative

In 1990 A. V. Borovik put to Kourovka notebook

Question 11.11 (part):

Is the Baer-Suzuki theorem true in the class of periodic groups?

V. D. Mazurov, A. Yu. Ol'shanskii, A. I. Sozutov (2015)

There is a group of a period divisible by 2^{48} in which any two involutions generate a 2 group, but the 2 radical is equal to 1.

Thus (for $p = 2$), among groups of a sufficiently large period, Borovik's question has a negative answer.

Baer-Suzuki theorem for periodic groups: positive

A. M. (2014):

Let G be a group of the period $n = 4k$, where k is odd, containing an involution i. If any two elements of i^G generate a 2-subgroup, then $\langle i^G \rangle$ is a group of period 4; in particular (by I.N.Sanov's theorem), it is locally finite .

J. Tang, N. Yang, A.M. (2024):

Let G be a periodic group with no elements of order 3^2. If $C \subset G$ is a normal subset and any pair of elements from C generates a 3-group, then $\langle C \rangle$ is a group of period 3; in particular, it is locally finite.

Baer-Suzuki theorem for periodic groups:summary

Question 11.11 (part):

Is the Baer-Suzuki theorem true in the class of periodic groups?

Thus, on question 11.11, a negative answer was received for $p = 2$ and a big period (2^{48});
the positive answer is
for $p = 2$ in groups without elements of the order of 8;
and for $p = 3$ in groups without elements of the order of 9:
that is, in all cases when the local finiteness of the corresponding group $\langle C \rangle$ will automatically follow from a known positive solution of the Burnside problem.

n-transposition groups

Let's consider another example where restrictions are imposed on subgroups generated by two elements from C, and the structure of $\langle C \rangle$ is studied.

G is n-**transposition group**, if it is generated by a normal set of involutions (elements of order 2) D such that
$\forall x, y \in D$ the order $|xy| \leq n$.

The most famous case is $n = 3$:
finite 3-transpositions groups were studied by B. Fischer, who discovered several new sporadic groups along the way.
Finite 3-transpositions groups are classified.
Methods are called "internal geometric analysis".

3-transposition groups

Theorem (B.Fisher, 1971)

Let (G, D) be a finite connected 3-transposition group with no solvable subgroups $\neq 1$, then G is one of the following:

- S_m, $m \geq 5$;
- $O^{\epsilon}_{2m}(2)$, $\epsilon = \pm$, $m \geq 3$, $(m, \epsilon) \neq (3, +)$;
- $Sp_{2m}(2)$, $m \geq 3$;
- ${}^{+}\Omega^{\epsilon}_{m}(3)$, $\epsilon = \pm$, $m \geq 6$;
- $SU_m(2)$, $m \geq 4$;
- Fi_{22}, Fi_{23}, Fi_{24}, $P\Omega^{+}_{8}(2) : S_3$, $P\Omega^{+}_{8}(3) : S_3$.

The set D is defined uniquely up to an automorphism of G.

In 1995 J.Hall and H.Cuypers improved the statement and proved that 3-transposition groups are **locally finite**.

Examples for $n > 3$

1. The sporadic group Baby Monster is a 4-transposition group.

2. The extension of the free Burnside group $B(2,5):2$ by an involution inverting generatos, is a group of 5-transpositions.
It is not known whether such a group is finite.

3. The largest sporadic group (Monster) is generated by three $6-$transpositions (from the conjugacy class 2A).

4. Miyamoto involutions of a Majorana algebra are 6-transpositions.

Timmesfeld-1975

Generalizations: $\{3,4\}^+$ (T-70,73), odd (A-72).
D is called a set of **root involutions** for G, if

- $G = \langle D \rangle$, $D = D^G$;
- $|xy| = 2, 4$ or odd for $x, y \in D$;
- if $|xy| = 4$ for $x, y \in D$, then $(xy)^2 \in D$.

Timmesfeld's classification, 1975)

Conditions: no normal solvable subgroups $\neq 1$, $G' = G''$. Classification: $G = \prod G_i$, where G_i:

- is simple in characteristic 2, different from ${}^2F_4(q)$;
- orthogonal $O_n^\epsilon(q)$, $q = 2^m$; $O_n^\epsilon(p)$, $p \in \{3, 5\}$;
- S_n or $L_2(q) \wr S_n$, $q = 2^m$;
- $SU_m(2)$, $m \geq 4$;
- A_6, HJ, Fi_{22}, Fi_{23}, Fi_{24}.

Other known results for small n

1. H. Cuypers, J. I. Hall: groups of 3-transpositions are locally finite.
2. S. Khasraw, J. McInroy, S. Shpectorov: groups generated by three 4-transpositions, are finite and known.
3. The following result is on groups, generated by three 6-transpositions, two of which commute.

V. A. Afanasev, A. M.(2024)

Let $G = \langle x, y, z \rangle$, where $x,\ y,\ z$ are 6-transpositions, with $|xy| = 2$ and $|xz| < 6$. Then G is a factor of one of the following groups: $l^2 : D_{12}$ or $l^2 : D_8$, with $l = 4, 5, 6$; $2^t : D_{2t}$, with $t = 5, 6$; $(S_4 \times S_4) : 2^2$; $(A_5 \times A_5) : 2^2$; $PGL(2, 9)$; $3^4 : (D_8 \times S_3)$; $2 \times (2^s : S_5)$, with $s = 4, 6$; $k^5 : (2^4 : D_{10})$, with $k = 2, 3$; $2^{10} : (2 \times PSL(2, 11))$; $O_2 : A_5$ with $|O_2| = 2^{10}$; $O_3 : D_{20}$ with $|O_3| = 3^8$; $2 \times M_{12}$; $(2.M_{22}) : 2$; $2 \times 2^5 : S_6$ or $2 \times 3.S_6$. In particular, G is finite.

Elements of order 3

What if we consider elements of order 3 (instead of involutions), and restrict subgroups generated by two of them?

Let $\mathfrak{M}$ be a set of groups. Let's say that G is $\mathfrak{M}$-group, if G is generated by a class D of conjugate elements of order 3 such that any pair of elements from D generates a subgroup, isomorphic to a factor of some group from $\mathfrak{M}$.

Example: alternating group $A_n = \langle (1, 2, 3)^{A_n} \rangle$ is $\mathfrak{M}_0 = \{3^2, A_4, A_5\}$-group:

it is generated by 3-cycles, and any two various 3-cycles either
have no common elements and so commute,
or contain two common elements and generate A_4,
or contain one common element and generate A_5.

Motivation in finite groups

(Old) observation: sporadic Conway group (Co_0), which is an automorphism group of 24-dim Leech lattice, is $\mathfrak{M}_3 = \{3^2, SL_2(3), SL_2(5)\}$ -group.

Results of Aschbacher and Hall

M. Aschbacher, M. Hall in 1973 published a paper describing **finite** $\mathfrak{M}_1 = \{3^2, SL_2(3)\}$-groups (in our terms).

M. Aschbacher, M. Hall(1973)

Let G be a finite group, generated by a conjugacy class D of subgroups of order 3, such that any pair of non-commuting subgroups in D generates a subgroup isomorphic to $SL_2(3)$ or A_4. Then D is isomorphic to $Sp_n(3), U_n(3)$ or $PGU_n(2)$.

Results of Stellmacher

B. Stellmacher in 1974 published a paper (in German), describing **finite** $\mathfrak{M}_2 = \{3^2, SL_2(3), A_5\}$-groups.

B. Stellmacher(1974)

Let G be a finite group satisfying the following conditions:
1. G is generated by a class D of elements of order 3. Two non-commuting elements from D generate a subgroup isomorphic to A_4, A_5, or $SL_2(3)$.
2. There is a pair of elements in D generating A_5.
3. $O_2(G) = Z(G) = 1$.

Then G is isomorphic to $Sp(2n, 2)$ for $n \geq 3$, $O^+(2n, 2)$ or $O^-(2n, 2)$ for $n \geq 3$, A_n for $n \geq 5$, HJ, $G_2(4)$, or Co_1.

Note that **series** of groups from the conclusion «appear to be» 3-transposition groups (A_n corresponds to S_n).

Further motivation

Note that if G is a of 3-transpositions of symplectic type, meaning that no three 3-transposition generate $3^2 : 2$ or $3^{1+2} : 2$,
then any pair of elements of order 3 in G, inverted by a 3-transposition either commutes or generates a subgroup isomorphic to A_4 or $SL_2(3)$ (if 4-generated diagram is the square, i.e. $W(D_4)$), or A_5 (when the diagram can be reduced to a line).
The simple way to verify it is by using Sozutov paper of 1992 describing rank 4 groups of symplecting type by a direct check.

Additional example

Example: $PSL_n(3)$ is a $\{3^{1+2}, SL_2(3)\}$-group. Indeed, let V be a

corresponding space and $t = t_1$ and t_2 be two transvections. By definition this means that $[V, t]$ is 1-dimensional and is contained in $C_V(t)$. Up to a symmetric one of the following holds.

1. $[V, t_1] \subseteq C_V(t_2)$ and $[V, t_2] \subseteq C_V(t_1)$. Then $\langle t_1, t_2 \rangle \simeq 3^2$.
2. $[V, t_1] \not\subseteq C_V(t_2)$ and $[V, t_2] \not\subseteq C_V(t_1)$. Then $\langle t_1, t_2 \rangle$ is contained in $SL([V, t_1] \oplus [V, t_2]) \simeq SL_2(3)$.
3. $[V, t_1] \subseteq C_V(t_2)$ and $[V, t_2] \not\subseteq C_V(t_1)$. Then $[t_1, t_2]$ is also a transvection. Hence in this case $\langle t_1, t_2 \rangle \simeq 3^{1+2}$.

Questions

Let's write out the classes mentioned above:
$\mathfrak{M}_0 = \{3^2, A_4, A_5\}$
$\mathfrak{M}_1 = \{3^2, SL_2(3)\}$
$\mathfrak{M}_2 = \{3^2, SL_2(3), A_5\}$
$\mathfrak{M}_3 = \{3^2, SL_2(3), SL_2(5)\}$
$\mathfrak{M}_e = \{3^{1+2}, SL_2(3)\}$
In this regard, it is interesting to introduce the class
$\mathfrak{M}_4 = \{3^{1+2}, SL_2(3), SL_2(5)\}$,
which contains all the classes listed above.

It is interesting to ask
Question i. Is $\mathfrak{M}_i$-group locally finite $(i = 0, \dots, 4)$?

Known results

For $i = 0$ the answer is known and in some sense it is the characterization of alternating groups.

V. Mazurov (2005)

A group G generated by a class D of conjugate elements of order 3, such that any two non-commuting elements of D generate a subgroup isomorphic to an alternating group of degree 4 or 5, is locally finite. More precisely, either G contains a normal elementary abelian 2-subgroup of index 3, or it is isomorphic to an alternating froup of some (possibly infinite) set.

For $i = 3, 4$ it is also interesting to obtain the description of finite $\mathfrak{M}_i$-groups.

Let's also mention an old result related to $\{3, SL_2(3), SL_2(5)\}$-groups.

A. Maximenko, A. M. (2007)

Let G be a group generated by a conjugacy class of elements of order 3, such that any two elements from the class generated a subgroup, isomorphic to $Z_3, A_4, A_5, SL_2(3)$ or $SL_2(5)$. Then either G is isomorphic to one of the groups $U_3(3), HJ, G_2(4), 2.HJ, 2.G_2(4)$, or G is an extension of a locally finite 2-group by a group of order 3, or G is an extension of a locally finite 2-group by a group isomorphic to A_5. In particular, G is locally finite.

$\mathfrak{M}_4$: 3-generated are classified

We started a systematic study with colleagues. A description of the subgroups generated by three elements from D is obtained. All of them turned out to be finite.
Depending on the configuration on the sides of the triangle (vertices are generative, edge labels are isomorphism classes of the corresponding 2-generated subgroups), all possible 3-generated groups are described.
This description allows us to use the «Fischer's approach» in the future: to attach a new generator to a known configuration.

Geometrical approach

Another well-known approach to the study of 3- transpositions is «geometric», which is also reflected in our work. I will give an appropriate example, where the Aschbacher-Hall reasoning is slightly modified.

Lemma *Assume* $\mathfrak{M}_1$*-group* G *contains a proper* D*-subgroup* $H \simeq U_3(3)$*. Then any* D*-element not in* H*, commutes with some* D*-element from* H*.*

The class in $U_3(3)$

$H \simeq U_3(3)$, there are 28 elements in x^H

$$\{x; y, x^y, y^x; z, x^z, z^x; (y^{zx})^{-1}, (y^z)^{-1}, (y^{zx^{-1}})^{-1}; (z^{yx^{-1}})^{-1}, (z^{yx})^{-1}, (z^y)^{-1}; z^{xyx}, z^{xy}, (x^{yz})^{-1}; (x^{zy})^{-1}, y^{xzx}, y^{xz}; (z^{xy^{-1}})^{-1}, x^{yz^{-1}}, (z^{xy^{-1}x})^{-1}; (x^{yz^{-1}y})^{-1}, (y^{xz^{-1}})^{-1}, x^{zy^{-1}}; (x^{yzyx})^{-1}, (x^{yzy})^{-1}, y^{zx^{-1}z}\}$$

and their inverses, no two D-subgroups commute, and any three elements, that are not in $SL_2(3)$, generate $U_3(3)$, and $\sim$ is not transitive.
Excluding x other $28 - 1 = 27$ listed D-elements of $U_3(3)$ are split in triples, so that with x they all lie in one $SL_2(3)$-subgroup.

Design

Professor Da Zhao's talk was on designs.
Consider a set of points consisting of D-subgroups, and call the block four D-subgroups lying in a common subgroup isomorphic to $SL_2(3)$. Thus we obtain $2-(28,4,1)$ design, or Steiner 2-design, which is finite incidence geometry, with 28 points and 63 lines, consisting of 4 points, such that any pair of points lies in exactly one block (or line).
G is 2-transitive on points.

Lemma If the D element is equivalent to three elements from the block, then it is equivalent to the fourth, or commutes with it.

Reduction of brute force

Lemma (we are proving) *Assume* $\mathfrak{M}_1$*-group* G *contains a proper* D*-subgroup* $H \simeq U_3(3)$*. Then any* D*-element not in* H*, commutes with some* D*-element from* H*.* Fix the generators x, y, z of $H \simeq U_3(3)$ (satisfying the corresponding defining relations) and assume the opposite. Up to changing x to y and t to t^{-1} we may assume that t is equiivalent to at least 16 of 28 elements and also $t \sim x$ (equivalent means that the relation $txt = xtx$ holds).
Then according to the helping Lemma, if t is equivalent to three elements from the block, then it is equivalent to the fourth. Therefore, each triple must have at least one element equivalent to t.
The properties of the $2-(28, 4, 1)$ design allow you to reduce the search to exactly two cases.