The Terwilliger algebra of the q-Johnson graph

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Genesis of Finite Simple Groups: 66 and 6, Shijiazhuang

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Contents

Definitions and notations

Let $\Gamma = (X, R)$ be a finite, simple, undirected, connected graph.

- Distance $\partial(x, y)$: the length of a shortest path connecting x and y.
- Diameter $D := D(\Gamma) = \max\{\partial(x, y) \mid x, y \in X\}$.
- For each $0 \leq i \leq D$, define the ith distance matrix A_i by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i. \end{cases}$$

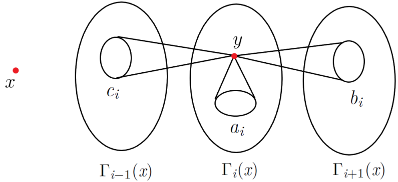
- $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ for a vertex x $(0 \leq i \leq D)$.
- Regular with valency k: $|\Gamma_1(x)| = k$ for all vertices in Γ.

Distance-regular graph (DRG)

A connected graph Γ is called distance-regular (DR) if there are constants a_i, b_i, c_i $(0 \le i \le D = D(\Gamma))$ s.t. for any $x, y \in X$, if $\partial(x, y) = i$ then

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|,$$
$$a_i = |\Gamma_i(x) \cap \Gamma_1(y)|,$$
$$b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|.$$

- Intersection numbers: a_i, b_i, c_i for $0 \le i \le D$ and $b_D = c_0 = 0$.
- $a_i + b_i + c_i = b_0 = k$.



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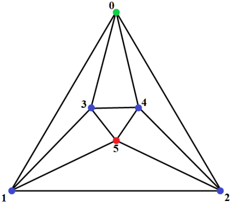
- Intersection numbers: a_i, b_i, c_i for $0 \leq i \leq D$ and $b_D = c_0 = 0$.
- $a_i + b_i + c_i = b_0 = k$.
- For example: Octahedron with $\{b_0, b_1; c_1, c_2\} = \{4, 1; 1, 4\}$.



Bose-Mesner algebra

Let Γ be a DRG. Let $A = A_1$.

- $AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$.
- Bose-Mesner algebra
  $\mathfrak{A} = \langle A \rangle = \mathrm{span}\{A^0, A^1, \cdots, A^D\} = \mathrm{span}\{A_0, A_1, \cdots, A_D\}$.
- $A_i = p_i(A)$, where p_i is a polynomial of degree i.
- The ordering $A_0, A_1, \cdots, A_D$ are called P-polynomial ordering.

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- The ordering $A_0, A_1, \cdots, A_D$ are called P-polynomial ordering.
- $A_i \circ A_j = \delta_{ij}A_i$, $\sum_{i=0}^{D} A_i = J$.
- Bose-Mesner algebra is closed under matrix product and Hadamard product.

Example

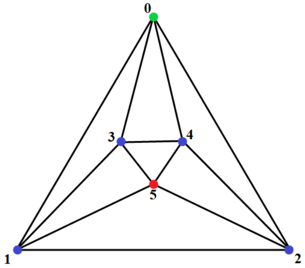

Figure: Octahedron

- $A_0 = I = A^0$;

$$A = A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{4}(A^2 - 2A - I).$$

- $\mathfrak{A} = \mathrm{span}\{A_0,\ A_1,\ A_2\} = \langle A \rangle$.

Q-polynomial DRG

Let Γ be a DRG. Let $A = A_1$.

- Let $E_i = \prod_{j=0, j\neq i}^{D} \frac{A-\theta_j I}{\theta_i - \theta_j}$, where $\theta_0, \theta_1, \cdots, \theta_D$ are distinct eigenvalues of A.
- E_i is the orthogonal projection onto the eigenspace V_i corresponding to θ_i.
- $E_i E_j = \delta_{ij} E_i$ and $\sum_{i=0}^{D} E_i = I$.
- $\mathfrak{A} = \mathrm{span}\{E_0, E_1, \cdots, E_D\}$.

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- $E_i E_j = \delta_{ij} E_i$ and $\sum_{i=0}^{D} E_i = I$.
- $\mathfrak{A} = \mathrm{span}\{E_0, E_1, \cdots, E_D\}$.
- $E_1 \circ E_i = \frac{1}{|X|}(b^*_{i-1} E_{i-1} + a^*_i E_i + c^*_{i+1} E_{i+1})$.
- that is equivalent to Q-polynomial property, i.e. $E_i = q_i(E_1)$ w.r.t. Hadamard product, where q_i is a polynomial of degree i.
- Dual intersection numbers: a^*_i, b^*_i, c^*_i. They are non-negative real numbers.
- The ordering $E_0, E_1, \cdots, E_D$ are called Q-polynomial ordering.

Example

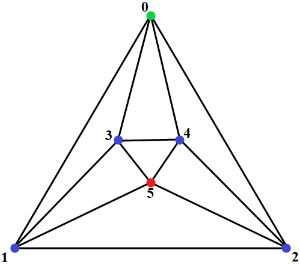

Figure: Octahedron

- Spec(Octahedron)$= \{4^1,\ 0^3,\ (-2)^2\}$.
- $E_0 = \frac{1}{6}J$ (J is all 1's matrix);

$$E_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$E_2 = \frac{1}{6}\begin{pmatrix} 2 & -1 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

- $\mathfrak{A} = \mathrm{span}\{E_0,\ E_1,\ E_2\}$.

From Hadamard product to the Dual Bose-Mesner Algebra

Example: For the octahedron, Bose-Mesner algebra contains

$$M = \begin{pmatrix} 3 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -2 \\ \frac{1}{2} & 3 & \frac{1}{2} & \frac{1}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 3 & -2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 & 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{1}{2} & \frac{1}{2} & 3 & \frac{1}{2} \\ -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 3 \end{pmatrix}, N = \begin{pmatrix} 5 & 4 & 4 & 4 & 4 & 0 \\ 4 & 5 & 4 & 4 & 0 & 4 \\ 4 & 4 & 5 & 0 & 4 & 4 \\ 4 & 4 & 0 & 5 & 4 & 4 \\ 4 & 0 & 4 & 4 & 5 & 4 \\ 0 & 4 & 4 & 4 & 4 & 5 \end{pmatrix},$$

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$$M \circ N = \begin{pmatrix} 15 & 2 & 2 & 2 & 2 & 0 \\ 2 & 15 & 2 & 2 & 0 & 2 \\ 2 & 2 & 15 & 0 & 2 & 2 \\ 2 & 2 & 0 & 15 & 2 & 2 \\ 2 & 0 & 2 & 2 & 15 & 2 \\ 0 & 2 & 2 & 2 & 2 & 15 \end{pmatrix}.$$

From Hadamard product to the Dual Bose-Mesner Algebra

Focus on the first column:

$$M = \begin{pmatrix} 3 & & & & & \\ \frac{1}{2} & & & & & \\ \frac{1}{2} & & & & & \\ \frac{1}{2} & & & & & \\ \frac{1}{2} & & & & & \\ -2 & & & & & \end{pmatrix}, \ N = \begin{pmatrix} 5 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 0 & & & & & \end{pmatrix}$$

$$M \circ N = \begin{pmatrix} 15 & & & & & \\ 2 & & & & & \\ 2 & & & & & \\ 2 & & & & & \\ 2 & & & & & \\ 0 & & & & & \end{pmatrix}$$

From Hadamard product to the Dual Bose-Mesner Algebra

Focus on the first column:

$$\begin{pmatrix} 3 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -2 \end{pmatrix} \circ \begin{pmatrix} 5 \\ 4 \\ 4 \\ 4 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 2 \\ 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

From Hadamard product to the Dual Bose-Mesner Algebra

This is the same operation as multiplication of diagonal matrices:

$$\begin{pmatrix} 3 & & & & & \\ & \frac{1}{2} & & & & \\ & & \frac{1}{2} & & & \\ & & & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \\ & & & & & -2 \end{pmatrix} \begin{pmatrix} 5 & & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 4 & \\ & & & & & 0 \end{pmatrix} = \begin{pmatrix} 15 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 0 \end{pmatrix}$$

So, if x_0 indexes the first column, for example, the dual Bose-Mesner algebra encodes this $\circ$ operation.

Dual Bose-Mesner algebra

Let $\Gamma = (X, R)$ be a Q-polynomial DRG. Fix a base point $x_0 \in \mathrm{X}$.

- Define $A_i^* = A_i^*(x_0)$ with $(A_i^*)_{xx} = |X|(E_i)_{xx_0}$.
- $A_1^* A_i^* = (b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^*)$.
- Dual Bose-Mesner w.r.t. x_0: $\mathfrak{A}^* = \mathfrak{A}^*(x_0) = \mathrm{span}\{A_0^*,\ A_1^*,\ \ldots,\ A_D^*\}$.

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- Define $E_i^* = E_i^*(x_0)$ with $(E_i^*)_{xx} = 1$ if $x \in \Gamma_i(x_0)$, 0 otherwise, where $\Gamma_i(x_0) = \{x \in X \mid \partial(x, x_0) = i\}$.
- $E_i^* E_j^* = \delta_{ij} E_i^*$ and $\sum_{i=0}^{D} E_i^* = I$.
- $\mathfrak{A}^* = \mathrm{span}\{E_0^*,\ E_1^*,\ \ldots,\ E_D^*\}$.

Terwilliger algebra (or subconstituent algebra)

Let $\Gamma = (X, R)$ be a Q-polynomial DRG. Fix a base point x_0.

- $T = T(x_0) = \langle \mathfrak{A}, \mathfrak{A}^* \rangle \subset \mathrm{Mat}_X(\mathbb{C})$ is the Terwilliger algebra w.r.t. x_0 over $\mathbb{C}$.
- $T = \langle A, E_i^* \mid 0 \leq i \leq D \rangle$.

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- $T = \langle A, E_i^* \mid 0 \le i \le D \rangle$.
- T is a semi-simple algebra.
- $V = \mathbb{C}^X \simeq \bigoplus_{x \in \mathrm{X}} \mathbb{C}x$: the standard module for T.
- T-submodule $W \subset V$ s.t. $TW \subseteq W$.
- T-submodule W is irreducible $\Longleftrightarrow W \neq 0,\ W$ does not properly contain a nonzero T-submodule.
- V is an orthogonal direct sum of irreducible T-submodules.
- V is a faithful T-module.
  In particular, every irreducible T-module appears in V.

Motivations

Let $\Gamma = (X, R)$ be a Q-polynomial DRG. Fix a base point x_0. Let $T = T(x_0)$ be the Terwilliger algebra of Γ. Let $V = \mathbb{C}^X$.

- $H = \mathrm{Aut}(\Gamma)_{x_0}$, the stabilizer of $\mathrm{Aut}(\Gamma)$ w.r.t. x_0.
- Centralizer algebra of H
  $S = \mathrm{Hom}_H(V, V) = \{f : V \to V \mid f(hv) = hf(v)$ for all $h \in H, v \in V\}$.
- T is a combinatorial analog of S.
- In general, $T \subseteq S$.
- Remark: $S = \mathrm{Hom}_H(V, V) \simeq \mathrm{Mat}_X(\mathbb{C})$ is a coherent algebra, i.e., closed under the matrix product and the Hadamard product, whereas T may not be closed under the Hadamard product.

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- List of Q-polynomail DRGs: Hamming graphs, Johnson graphs, q-Johnson graphs, dual polar graphs, bilinear form graphs, classical form graphs, exceptions.
- Question 1: Which graphs in the list satisfy $T = S$?

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- Question 2: If $T \subset S$ holds, how large is the gap between S and T?
  $T \circ T = S$?

Related Results
Hamming graph

- Let $q \geq 2$, $D \geq 1$ be integers.
- $\Omega = \{0, 1, 2, \ldots, q-1\}$.
- Hamming graph $H(D, q)$ has vertex set $X = \Omega^D$.
- $x \sim y$ if they differ in exactly one position.
- $H(D, 2)$ is D-cube.
- An undelying space for coding theory.
- $T = S$ holds[a] .

[a] D. Gijswijt, A. Schrijver, and H. Tanaka. "New upper bounds for nonbinary codes based on the Terwilliger algebra and semidefinite programming". In: *J. Combin. Theory Ser. A* 113.8 (2006), pp. 1719–1731.

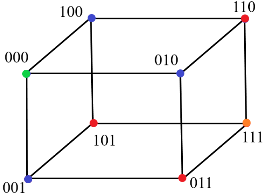

Figure: $H(3, 2)$

Related Results
Johnson graph

- Let $1 \leq D \leq N$ be integers.
- $\Omega = \{1, 2, \ldots, N\}$.
- Johnson graph $J(N, D)$ has vertex set $X = \binom{\Omega}{D}$.
- $x \sim y$ if $|x \cap y| = D - 1$.
- $J(N, D) \simeq J(N, N - D)$.
- Usually, assume $N \geq 2D$. Diameter $= D$.
- An undelying space for design theory.

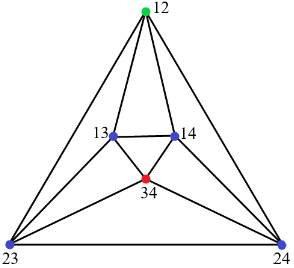

Figure: $J(4, 2)$

[1] Y-Y. Tan et al. "The Terwilliger algebra of the Johnson scheme $J(N, D)$ revisited from the viewpoint of group representations". In: *European J. Combin.* 80 (2019), pp. 157–171.

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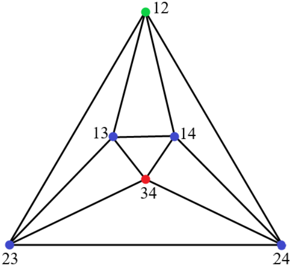

Figure: $J(4, 2)$

- $T = S$ holds when $N \neq 2D$;
- $T \subset S$ holds when $N = 2D$.[1]

[1] Y-Y. Tan et al. "The Terwilliger algebra of the Johnson scheme $J(N, D)$ revisited from the viewpoint of group representations". In: *European J. Combin.* 80 (2019), pp. 157–171.

Related Results
q-Johnson graph

- Let q be a prime power and $\mathbb{F}_q$ be a finite field.
- $1 \leq D \leq N$ be integers.
- $\Omega = \mathbb{F}_q^N$.
- q-Johnson graph $J_q(N, D)$ has vertex set $X = \binom{\Omega}{D}_q$.
- $x \sim y$ if $\dim(x \cap y) = D - 1$.
- $J_q(N, D) \simeq J_q(N, N - D)$.
- Usually, assume $N \geq 2D$. Diameter $= D$.
- $J_q(N, D)$ is a q-analog of $J(N, D)$.

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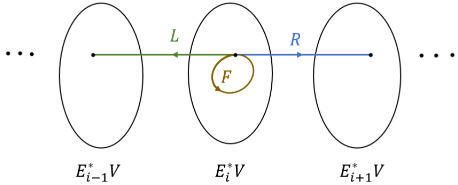
- $J_q(N, D)$ is a q-analog of $J(N, D)$.
- $T \subset S$ holds.

Decomposition of A

Let $\Gamma = (X, R)$ be a Q-polynomial DRG. Let $V = \mathbb{C}^X$.

- $T = T(x_0) = \langle A, E_i^* \mid 0 \le i \le D \rangle$: the Terwilliger algebra w.r.t x_0.
- Lowering map $L = L(x_0) = \sum_{i=1}^{D} E_{i-1}^* A E_i^*$ and $L E_i^* V \subseteq E_{i-1}^* V$;
- Flat map $F = F(x_0) = \sum_{i=0}^{D} E_i^* A E_i^*$ and $F E_i^* V \subseteq E_i^* V$;
- Raising map $R = R(x_0) = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*$ and $R E_i^* V \subseteq E_{i+1}^* V$.
- $A = L + F + R$.



Parameters of an irreducible T-module

Let W be an irreducible T-submodule of V for Γ.

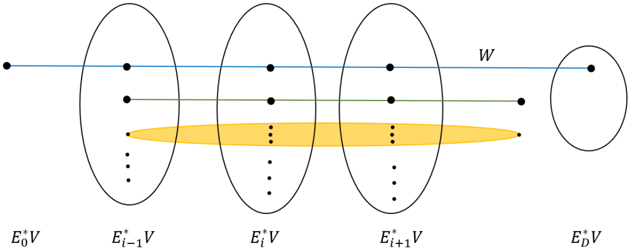
- endpoint of W: $\nu := \min\{i \mid 0 \le i \le D,\ E_i^* W \ne 0\}$.
- dual-endpoint of W: $\mu := \min\{i \mid 0 \le i \le D,\ E_i W \ne 0\}$.
- diameter of W: $d = \#\{i \mid E_i^* W \ne 0\} - 1 = \#\{i \mid E_i W \ne 0\} - 1$.
- W is thin $\Leftrightarrow \dim E_i^* W \le 1 \ \Leftrightarrow \dim E_i W \le 1$ for all i.

Terwilliger, 1993

For the known Q-polynomial DRG, **thin** irreducible T-modules are determined by parameters (ν, μ, d, e), where e is a auxiliary parameter.[a]

[a] P. Terwilliger. "The subconstituent algebra of an association scheme III". In: *J. Algebraic Combin.* 2 (1993), pp. 177–210.

Parameters of an irreducible T-module



Irreducible T-modules for the Johnson graph[2]

Let $\Gamma = J(N, D)$. Let W be an irreducible T-module with $d \geq 1$.

- W is thin.
- W is determined by (ν, μ, d) (up to isomorphism) that satisfy

$$(\Delta_1) \begin{cases} 0 \leq \frac{D-d}{2} \leq \nu \leq \mu \leq D - d \leq D \\ d \in \{D - 2\nu, \ \min\{D - \mu, N - D - 2\nu\}\} \end{cases}$$

[2] P. Terwilliger. "The subconstituent algebra of an association scheme III". In: *J. Algebraic Combin.* 2 (1993), pp. 177–210.

Irreducible T-modules for the q-Johnson graph[3]

Let $\Gamma = J_q(N, D)$. Let W be an irreducible T-module with $d \geq 1$.

- W is thin.
- W is determined by (ν, μ, d, e) (up to isomorphism) that satisfy

$$(\Delta_2)\begin{cases} 0 \leq \frac{D-d}{2} \leq \nu \leq \mu \leq D-d \leq D \\ e+d+D \text{ is even}, \quad |e| \leq 2\nu - D + d \\ d \in \{e+D-2\nu, \ \min\{D-\mu, e+D-2\nu+2(N-2D)\}\} \end{cases}$$

[3] P. Terwilliger. "The subconstituent algebra of an association scheme III". In: *J. Algebraic Combin.* 2 (1993), pp. 177–210.

Remark

- No proof is given in Terwilliger's paper, 1993.
- It is not clear whether all (ν, μ, d) in (Δ_1) for Johnson graph (all (ν, μ, d, e) in (Δ_2) for q-Johnson graph) appear from irreducible T-modules with $d \geq 1$.

Our work

We showed that all (ν, μ, d) with (Δ_1) for Johnson graph[a] (all (ν, μ, d, e) with (Δ_2) for q-Johnson graph[b]) appear from irreducible T-modules, including $d = 0$!

[a] Y-Y. Tan et al. "The Terwilliger algebra of the Johnson scheme $J(N, D)$ revisited from the viewpoint of group representations". In: *European J. Combin.* 80 (2019), pp. 157–171.

[b] X. Liang, T. Ito, and Y.Watanabe. "The Terwilliger algebra of the Grassmann scheme $J_q(N, D)$ revisited from the viewpoint of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$". In: *Linear Algebra Appl.* 596 (2020), pp. 117–144.

New parameters of an irreducible T-module for $J(N, D)$

Let Λ_1 be the set of ordered pairs (α, β) of non-negative integers $\alpha,\ \beta$ such that

$$(\Lambda_1)\begin{cases}0 \leq \alpha \leq \frac{D}{2},\\ 0 \leq \beta \leq \min\{D, \frac{N-D}{2}\},\\ 0 \leq \alpha + \beta \leq D.\end{cases}$$

Define a mapping $\varphi_1 : \Lambda_1 \to \Delta_1, (\alpha, \beta) \longmapsto (\nu, \mu, d)$ by

$$\nu = \max(\alpha, \beta),$$
$$\mu = \alpha + \beta,$$

$$d = \begin{cases} D - 2\alpha & \text{if } \beta - \alpha \leq 0,\\ D - \alpha - \beta & \text{if } 0 \leq \beta - \alpha \leq N - 2D,\\ N - D - 2\beta & \text{if } N - 2D \leq \beta - \alpha.\end{cases}$$

New parameters of an irreducible T-module for $J(N, D)$

Theorem (Liang, T.Ito, Y-Y.Tan, 2017)

(1) *The mapping* $\varphi_1 : \Lambda_1 \to \Delta_1$ *is a bijection if* $N \neq 2D$.

(2) *Let* $\bar{\Lambda}_1$ *be the subset of* Λ_1 *consisting of* $(\alpha, \beta) \in \Lambda_1$ *that satisfy* $\beta - \alpha \leq 0$. *Then the mapping* $\varphi|_{\bar{\Lambda}_1} : \bar{\Lambda}_1 \to \Delta_1$ *is a bijection if* $N = 2D$.

The meaning of Λ_1 from the viewpoint of group representations

Let $\Gamma = J(N, D)$.

- Let $\Omega = \Omega_1 \cup \Omega_2$, where $|\Omega_1| = D$, $|\Omega_2| = N - D$.
- $H = \mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2)$.
- $\mathrm{Sym}(\Omega_1)$ acts on $\binom{\Omega_1}{D-i}$, $\mathrm{Sym}(\Omega_2)$ acts on $\binom{\Omega_2}{i}$.
- Let $\pi_i^{(1)}$ be the permutation character of $\mathrm{Sym}(\Omega_1)$ on $\binom{\Omega_1}{D-i} \simeq \binom{\Omega_1}{i}$;
- Let $\pi_i^{(2)}$ be the permutation character of $\mathrm{Sym}(\Omega_2)$ on $\binom{\Omega_2}{i}$.
- $\pi_i^{(1)} = \sum_{\alpha=0}^{i} \chi_\alpha^{(1)}$, where $\chi_\alpha^{(1)}$ is an irreducible character of $\mathrm{Sym}(\Omega_1)$.
- $\pi_i^{(2)} = \sum_{\beta=0}^{i} \chi_\beta^{(2)}$, where $\chi_\beta^{(2)}$ is an irreducible character of $\mathrm{Sym}(\Omega_2)$.
- $\chi_\alpha^{(1)}\chi_\beta^{(2)}$: irreducible character of H, where $0 \le \alpha \le \min\{i, D-i\}$, $0 \le \beta \le \min\{i, N-D-i\}$.

The meaning of Λ_1 from the viewpoint of group representations

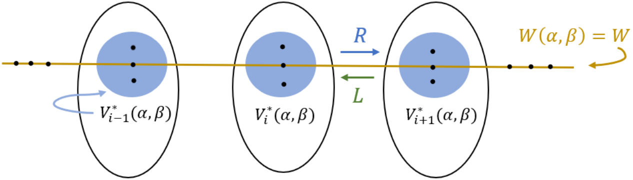
$V = \mathbb{C}^X$.

- $V_i^* = E_i^* V$ is a permutation H-module. H is multiplicity free on V_i^*.
- $V_i^* = \bigoplus_{\alpha,\beta} V_i^*(\alpha, \beta)$, where $V_i^*(\alpha, \beta)$ be the irreducible H-submodule with character $\chi_\alpha^{(1)} \chi_\beta^{(2)}$.
- $V = \bigoplus_i V_i^* = \bigoplus_i \bigoplus_{\alpha,\beta} V_i^*(\alpha, \beta) = \bigoplus_{\alpha,\beta} \bigoplus_i V_i^*(\alpha, \beta)$.
- $V^*(\alpha, \beta) = \bigoplus_i V_i^*(\alpha, \beta)$: the sum of irreducible H-submodules of V that afford the irreducible character $\chi_\alpha^{(1)} \chi_\beta^{(2)}$ of H, i.e., the homogeneous component of V that belongs to the isomorphism class $(\alpha, \beta) \in \Lambda$.
- $V = \bigoplus_{\alpha,\beta} V^*(\alpha, \beta)$.

The meaning of Λ_1 from the viewpoint of group representations

$S = \mathrm{Hom}_H(V, V)$.

- $V^*(\alpha, \beta)$ is invariant under S.
- Let $W \subseteq V$ be an irreducible S-submodule, then $\exists\,!\ (\alpha, \beta) \in \Lambda_1$ s.t. $W \subseteq V^*(\alpha, \beta)$.
- Let $W, W' \subseteq V^*(\alpha, \beta)$ be any irreducible S-submodules. Then $W \simeq W'$ as an S-submodule.
- This means that irreducible S-submodules W of V are parameterized by $(\alpha, \beta) \in \Lambda_1$, up to isomorphism.
- Let $W(\alpha, \beta)$ be an irreducible S-module.
- $R, L, F \in S$.



The meaning of Λ_1 from the viewpoint of group representations

Case $N \neq 2D$

- Irreducible T-submodules W of V with $d \geq 1$ are parameterized by $(\alpha, \beta) \in \Lambda_1$, up to isomorphism.
- For $W, W' \subseteq V$ with $d = d' \geq 1$,

$$W \simeq W' \text{ as } S\text{-modules} \Leftrightarrow W \simeq W' \text{ as } T\text{-modules.}$$

- In the case of $d = 0$, isomorphism classes of W is parameterized by (ν, μ).
- And $(\alpha, \beta) \to (\nu, \mu)$ is 1:1.
- $T = S$.

Case $N = 2D$

- $(\alpha, \beta) \to (\nu, \mu, d)$ is 2:1.
- $T \subset S$.

New parameters of an irreducible T-module for $J_q(N, D)$

Let Λ_2 be the set of ordered pairs (α, β, ρ) of non-negative integers α, β, ρ such that

$$(\Lambda_2) \begin{cases} 0 \leq \alpha \leq \frac{D-\rho}{2}, \\ 0 \leq \beta \leq \frac{N-D-\rho}{2}, \\ 0 \leq \alpha + \beta \leq D - \rho. \end{cases}$$

Define a mapping $\varphi_2 : \Lambda_2 \to \Delta_2, (\alpha, \beta, \rho) \longmapsto (\nu, \mu, d, e)$ by

$$\nu = \rho + \max\{\alpha, \beta\},$$
$$\mu = \rho + \alpha + \beta,$$

$$d = \begin{cases} D - \rho - 2\alpha & \text{if } \beta - \alpha \leq 0, \\ D - \rho - \alpha - \beta & \text{if } 0 \leq \beta - \alpha \leq N - 2D, \\ N - D - \rho - 2\beta & \text{if } N - 2D \leq \beta - \alpha. \end{cases}$$

$$e = \begin{cases} \rho & \text{if } \beta - \alpha \leq 0, \\ \rho + \alpha - \beta & \text{if } 0 \leq \beta - \alpha \leq N - 2D, \\ \rho - N + 2D & \text{if } N - 2D \leq \beta - \alpha. \end{cases}$$

New parameters of an irreducible T-module for $J_q(N, D)$

Theorem (Liang, T.Ito, Y. Watanabe, 2020)

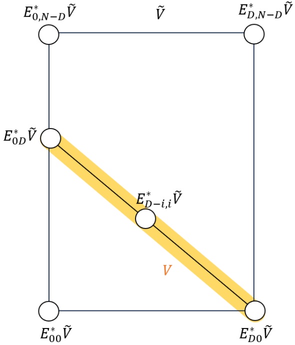
(1) *The mapping* $\varphi_2 : \Lambda_2 \to \Delta_2$ *is a bijection if* $N \neq 2D$.

(2) *Let* $\bar{\Lambda}_2$ *be the subset of* Λ_2 *consisting of* $(\alpha, \beta, \rho) \in \Lambda_2$ *that satisfy* $\beta - \alpha \leq 0$. *Then the mapping* $\varphi|_{\bar{\Lambda}_2} : \bar{\Lambda}_2 \to \Delta_2$ *is a bijection if* $N = 2D$.

The meaning of Λ_2 from the viewpoint of quantum affine algebras

Let $\Gamma = J_q(N, D)$.

- Let $\Omega = \mathbb{F}_q^N = \Omega_1 \oplus \Omega_2$, where $\Omega_1 = \mathbb{F}_q^D$, $\Omega_2 = \mathbb{F}_q^{N-D}$.
- $\widetilde{X} = \bigcup_{k=0}^{N} \binom{\Omega}{k}_q$: the set of all subspaces of Ω.
- Fix $x_0 = \Omega_1 \in \binom{\Omega}{D}_q$.
- For $0 \leq i \leq D, 0 \leq j \leq N-D$, let $\widetilde{X}_{ij} = \{x \in \widetilde{X} \mid \dim(x \cap x_0) = i, \dim x = i + j\}$.
- Define $E^*_{ij} = E^*_{ij}(x_0)$ by $(E^*_{ij})_{xx} = 1$ if $x \in \widetilde{X}_{ij}$, 0 otherwise.
- $\widetilde{V} = \mathbb{C}^{\widetilde{X}} = \bigoplus_{i,j} E^*_{i,j}\widetilde{V}$.
- $V = \mathbb{C}^X = \bigoplus_{i} E^*_{D-i,i}\widetilde{V} \subset \widetilde{V}$.



The meaning of Λ_2 from the viewpoint of quantum affine algebras

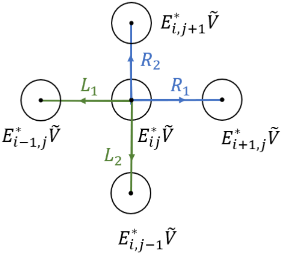
Let $\Gamma = J_q(N, D)$.

- Define $L_1,\ L_2,\ R_1,\ R_2 \in \mathrm{Hom}(\widetilde{V}, \widetilde{V})$ by

$$L_1 x = \sum_{\substack{y \in \widetilde{X}_{i-1,j} \\ x \text{ covers } y}} y, \qquad L_2 x = \sum_{\substack{y \in \widetilde{X}_{i,j-1} \\ x \text{ covers } y}} y,$$

$$R_1 x = \sum_{\substack{y \in \widetilde{X}_{i+1,j} \\ y \text{ covers } x}} y, \qquad R_2 x = \sum_{\substack{y \in \widetilde{X}_{i,j+1} \\ y \text{ covers } x}} y,$$

  where for $x, y \in \widetilde{X}$, x covers y iff $x \supset y$ and $\dim x = \dim y + 1$.
- $\mathcal{H} = \langle L_1, L_2, R_1, R_2, E^*_{ij} \mid \forall i, j\rangle \subseteq \mathrm{Mat}_{\widetilde{X}}(\mathbb{C})$.



The meaning of Λ_2 from the viewpoint of quantum affine algebras

Theorem (Y. Watanabe, 2017)

There is an algebra homomorphism from the quantum affine algebra $U_{\sqrt{q}}(\widehat{\mathfrak{sl}}_2)$ to algebra $\mathcal{H}$, and $\mathcal{H}$ is generated by its image together with the center.[a]

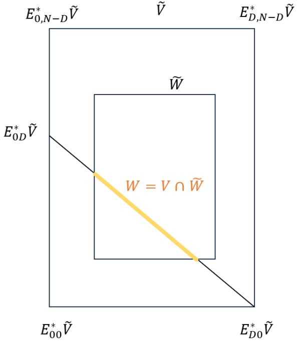
[a] Y. Watanabe. "An algebra associated with a subspace lattice over a finite field and its relation to the quantum affine algebra $U_q(\widehat{sl}_2)$". In: *J. Alg.* 489 (2017), pp. 475–505.

Theorem (Y. Watanabe 2017, X. Liang -T. Ito -Y. Watanabe 2020)

Irreducible $\mathcal{H}$-submodules $\widetilde{W}$ of $\widetilde{V}$ with are parameterized by $(\alpha, \beta, \rho) \in \widetilde{\Lambda}_2$, up to isomorphism, where $\widetilde{\Lambda}_2 = \{(\alpha, \beta, \rho) \mid 0 \leq \alpha \leq \frac{D-\rho}{2}, 0 \leq \beta \leq \frac{N-D-\rho}{2}\}$

The meaning of Λ_2 from the viewpoint of quantum affine algebras

- Recall $\Lambda_2 = \{(\alpha, \beta, \rho) \in \widetilde{\Lambda}_2 \mid 0 \leq \alpha + \beta \leq D - \rho\}$.
- If $\widetilde{W} \subseteq \widetilde{V}$ is an irreducible $\mathcal{H}$-module of type $\lambda = (\alpha, \beta, \rho) \in \Lambda_2$, then $W = \widetilde{W} \cap V$ is an irreducible T-module of type $\varphi_2(\lambda) \in \Delta_2$;
- If $\varphi_2(\lambda) = (\nu, \mu, 0, e) \in \Delta_2$, then $W = \widetilde{W} \cap V$ has diameter 0 and type (ν, μ).
- Conversely, if $W \subseteq V$ is an irreducible T-module with $d \geq 1$, then $\widetilde{W} = \mathcal{H}W \subseteq \widetilde{V}$ is an irreducible $\mathcal{H}$-module when $N \neq 2D$.



The meaning of Λ_2 from the viewpoint of group representations

Let $\Gamma = J_q(N, D)$.

- Let $\Omega = \mathbb{F}_q^N = \Omega_1 \oplus \Omega_2$, where $\Omega_1 = \mathbb{F}_q^D$, $\Omega_2 = \mathbb{F}_q^{N-D}$.
- Let $G = GL(N, q)$. G acts on $\binom{\Omega}{k}_q$ as automorphisms.
- Remark: If $N \neq 2D$, Aut$(\Gamma) = \mathrm{Gal}(\mathbb{F}_q)PGL(N, q) = P\Gamma L(N, q)$;
  If $N = 2D$, Aut$(\Gamma) = P\Gamma L(N, q).2$.
- Choose $x_0 = \Omega_1 \in \binom{\Omega}{D}_q$
- $H = G_{x_0} = (GL(D, q) \times GL(N-D, q)) \ltimes \mathrm{Mat}_{D\times(N-D)}(q)$.

Dunkl-Watanabe's dulity (X. Liang, T. Ito, in preparation)

$$\mathrm{Hom}_H(\widetilde{V}, \widetilde{V}) = \mathcal{H}.$$

- Irreducible H-modules of $\widetilde{V}$ are parameterized by (α, β, ρ) in $\widetilde{\Lambda}_2$.
- Irreducible H-modules of V are parameterized by (α, β, ρ) in Λ_2.

The meaning of Λ_2 from the viewpoint of group representations

Let $\Gamma = J_q(N, D)$.

- $V_i^* = E_{D-i,i}^* V$: H-submodule of V.
- $V_i^* = \bigoplus_{(\alpha,\beta,\rho)\in\Lambda_2} V_i^*(\alpha,\beta,\rho)$, where $V_i^*(\alpha,\beta,\rho)$: irreducible H-submodule.
- $$V = \bigoplus_{0\le i\le D} V_i^* = \bigoplus_{0\le i\le D} \bigoplus_{(\alpha,\beta,\rho)\in\Lambda_2} V_i^*(\alpha,\beta,\rho)$$
  $$= \bigoplus_{(\alpha,\beta,\rho)\in\Lambda_2} \bigoplus_{0\le i\le D} V_i^*(\alpha,\beta,\rho).$$
- Let $V^*(\alpha,\beta,\rho) = \bigoplus_i V_i^*(\alpha,\beta,\rho)$.
- $$V = \bigoplus_{(\alpha,\beta,\rho)\in\Lambda_2} V^*(\alpha,\beta,\rho)$$
  the homogeneous component decomposition of V as a H-module.
- $=$ the homogeneous component decomposition of V as an S-module.

The meaning of Λ_2 from the viewpoint of group representations

Let $\Gamma = J_q(N, D)$.

- Let $W \subseteq V$ be an irreducible S-submodule, then $\exists\,!\ (\alpha, \beta, \rho) \in \Lambda_2$ s.t. $W \subseteq V^*(\alpha, \beta, \rho)$.
- Let $W, W' \subseteq V^*(\alpha, \beta, \rho)$ be any irreducible S-modules. Then $W \simeq W'$ as S-module.
- This means that irreducible S-submodules W of V are parameterized by $(\alpha, \beta, \rho) \in \Lambda_2$, up to isomorphism.

The meaning of Λ_2 from the viewpoint of group representations

Case $N \neq 2D$.

- Irreducible T-submodules W of V with $d \geq 1$ are parameterized by $(\alpha, \beta, \rho) \in \Lambda_2$, up to isomorphism.
- For $W, W' \subseteq V$ with $d = d' \geq 1$,

$$W \simeq W' \text{ as } S\text{-modules} \Leftrightarrow W \simeq W' \text{ as } T\text{-modules.}$$

- In the case of $d = 0$, isomorphism classes of W is parameterized by (ν, μ).
- But $(\alpha, \beta, \rho) \to (\nu, \mu)$ is not 1:1, because the parameter e is dropped.
- $T \subset S$.

Case $N = 2D$.

- $(\alpha, \beta, \rho) \to (\nu, \mu, d, e)$ is 2:1.
- $T \subset S$.

Further problems

- For Johnson graphs, $T \circ T = S$ when $N = 2D$?
- For q-Johnson graphs, $T \circ T = S$?
- What are the relations between T and S for dual polar graphs, bilinear form graphs, ...

Thank you!