On distance-regular graphs with classical parameters

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(Based on joint work with Hongjun Ge, Chenhui Lv, Qianqian Yang. Some of it is still ongoing.)

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Outline

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1. Distance-regular graphs
   - Definitions
   - DRG with classical parameters
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   - Known classification results
3. Bounds on α and β
   - Bounds on α
   - Bounds on β
   - Tools for the proof

Notations

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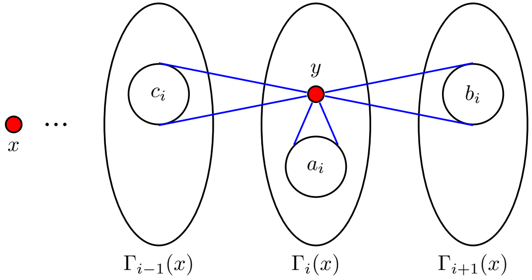
- $d(x, y)$:the distance between x and y.
- $D(\Gamma)$: diameter of Γ, if Γ is connected.
- $\Gamma_i(x)$= $\{y \mid d(x, y) = i\}$, $\Gamma(x) = \Gamma_1(x) = \{y \mid x \sim y\}$.
- The subgraph induced on $\Gamma(x)$ is the local graph of Γ at x, denoted by $\Delta(x)$.

Distance-regular graphs (DRG)

A connected graph Γ is called distance-regular (DRG) if there are numbers a_i, b_i, c_i, $0 \le i \le D(\Gamma)$, such that for any two vertices x and y with $d(x,y) = i$,

$$|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = c_i,\ |\Gamma_1(y) \cap \Gamma_i(x)| = a_i,\ |\Gamma_1(y) \cap \Gamma_{i+1}(x)| = b_i.$$

a_i, b_i, c_i, $0 \le i \le D(\Gamma)$ are called the intersection numbers of Γ.



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$$b_0 = c_i + a_i + b_i.$$

For a DRG Γ with diameter D, its intersection array is

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

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We say that a distance-regular graph Γ of diameter D has *classical parameters* (D, b, α, β) if the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), \tag{1}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \tag{2}$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geq 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b = 0$.

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where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geq 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b = 0.$
We note that $b \neq 0, -1$ by the following result.

Lemma

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and the diameter $D \geq 3$. Then, b is an integer such that $b \neq 0, -1$.

There are many examples of DRG with classical parameters namely:

- Hamming graphs and Doob graphs,
- Johnson graphs,
- Grassmann graphs and twisted Grassmann graphs,
- bilinear forms graphs,
- sesquilinear forms graphs,
- quadratic forms graphs,
- dual polar graphs,
- the Ustimenko graphs,
- the Hemmeter graphs.

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- An important subproblem of Bannai's problem is to classify the DRG with classical parameters, as they are Q-polynomial.
- This is a very hard problem as the twisted Grassmann graphs do exist.
- All the known infinite families of DRG with valency at least three and with unbounded diameter have classical parameters or are very closely related to an infinite family of DRG with classical parameters, like the folded hypercubes and the doubled Grassmann graphs.

- If b is negative then they are essentially classified by C.-W. Weng. There is still one infinite family of feasible parameter sets, for which we do not have any idea whether they exist or not.

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- Terwilliger in the 1980's classified the DRG with classical parameters with $b = 1$.
- He obtained:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) where $b = 1$ and $D \geq 4$. Then Γ is a Hamming graph, a halved cube, a Johnson graph or a Doob graph.

Outline

The Grassmann graphs

Theorem (Metsch(1995))

The Grassmann graphs $J_q(n, D)$ ($n \geq 2D$) are characterized by their intersection array if $n \geq \max\{2D+2, 2D+6-q\}$ and $D \geq 3$.

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- What happens for $n = 2D, n = 2D + 1$?
- Van Dam and K. (2005) found the twisted Grassmann graphs. They have the same parameters as $J_q(2D + 1, D)$, so the Grassmann graph $J_q(2D+1, D)$ is not determined by its intersection array.

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- He uses the vanishing Krein parameters to obtain some extra conditions on the c_2-graph.

Bilinear forms graphs

Metsch, building on earlier work of Sprague, Ray-Chaudhuri, Huang and Cuypers, showed:

Theorem (Metsch (1999))

The bilinear forms graph $\text{Bil}(D \times e, q)$ is characterized by its intersection array if $q = 2$ and $e \geq D + 4$ or $q \geq 3$ and $e \geq D + 3$.

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Can we obtain a bound of α in terms of b? We are going to talk about this in this talk.

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- We wonder whether this is the right bound.
- If $\alpha = 0$, $\beta \neq 0$, $b \geq 2$, $D \geq 4$ and locally the disjoint union of cliques, the graph must be a dual polar graph.
- This result is based on work by Brouwer and Wilbrink, De Bruyn, Cameron, Cohen and others.

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For $b = 2$, this shows that we have only 7 choices for α. Several of them can be removed by looking at the integrability of p^{2i}_{ii} for some i. This is still work in progress with H. Ge, C. Lv, and Q. Yang.

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C is something like b^5.

Outline

Partial linear spaces

- An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively.

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- If $(p, \ell) \in \mathcal{I}$ we say that p is *incident* with ℓ, or that p is on the line ℓ. The *order of a point* is the number of lines it is incident with and similarly for lines.

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- If $(p, \ell) \in \mathcal{I}$ we say that p is *incident* with ℓ, or that p is on the line ℓ. The *order of a point* is the number of lines it is incident with and similarly for lines.
- The *point-line incidence matrix* of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the $|\mathcal{P}| \times |\mathcal{L}|$-matrix such that the (p, ℓ) is 1 if p is incident with ℓ and 0 otherwise.

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- Let $X = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space. For a point p, define $\tau(p) :=$ the number of lines through p.
- We define $\tau(X)$ for a partial linear space $X = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, as $\tau(X) = \max\{\tau(x) \mid x \in \mathcal{P}\}$.

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- We define $\tau(X)$ for a partial linear space $X = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, as $\tau(X) = \max\{\tau(x) \mid x \in \mathcal{P}\}$.
- The point graph Γ of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the graph with vertex set $\mathcal{P}$ and two distinct points are adjacent if they are on a common line. Note that lines are cliques in Γ.

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Theorem

Let Γ be a DRG. Assume that there exists a positive integer s such that the following two conditions are satisfied:

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- $a_1+1 > (c_2-1)(2s-1)$.

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Define a line as a maximal clique with at least $a_1+2-(c_2-1)(s-1)$ vertices. Then $X=(V(\Gamma),\mathcal{L},\in)$ is a partial linear space, where $\mathcal{L}$ is the set of all lines, and Γ is the point graph of X. Moreover, every vertex is in at most s lines.

- For a DRG with classical parameters (D, b, α, β) with $b \geq 2$ and $D \geq 3$, the result of Metsch means that if $\beta > b^{D+5}$, then the graph is the point graph of partial linear space with large lines with $s \leq \frac{3}{2}b^D$.

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- Although a twisted Grassmann graph is the point graph of a partial linear space, its lines are not the maximum cliques. The twisted Grassmann graphs have $\beta > b^D$.

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- A clique with equality in the lemma is called a *Delsarte clique*.
- A DRG Γ is called *geometric* if it is the point graph of a partial linear space with Delsarte cliques as its lines. This is equivalent that we can partition the edge set of Γ into Delsarte cliques.

Using the method of Metsch, he used in his bilinear forms graph paper in 1999, with some modifications and simplifications, we were able to show:

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Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$.

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The C_1 is something like b^6 and the twisted Grassmann graphs are not geometric.

- With some extra work we were able to show that $\beta \leq C_1 b^D$.
- **Thank you for your attention.**