Axial algebras of Monster type

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2nd, December 2024

Set up

Let $\mathbb{F}$ be a field and let A be a commutative $\mathbb{F}$-algebra. As usual, the adjoint map associated to an element $a \in A$ is the map

$$\mathrm{ad}_a : A \to A, \;\; x \mapsto ax$$

For $\lambda \in \mathbb{F}$, define

$$A_\lambda(a) := \{v \in A \mid av = \lambda v\}$$

Let $\alpha, \beta \in \mathbb{F} \setminus \{0, 1\}$, with $\alpha \neq \beta$, and set

$$\mathcal{F} = \{0, 1, \alpha, \beta\}.$$

and let $\star : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ be the fusion law defined by the table

$\star$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$0, 1, \alpha$

$\mathcal{F}$-axes

Let A and $\mathcal{F}$ be as above.

Definition

An element $a \in A$ is called an $\mathcal{F}$-axis if

1. $a^2 = a$
2. ad_a *is semisimple with spectrum contained in* $\mathcal{F}$
3. *for all* $\lambda, \mu \in \mathcal{F}$,
$$A_\lambda(a)A_\mu(a) \subseteq \bigoplus_{\nu \in \lambda \star \mu} A_\nu(a),$$
that is, the product of a λ-eigenvector and a μ-eigenvector of ad_a is the sum of δ-eigenvectors, with $\delta \in \lambda \star \mu$.

Moreover, a is called primitive if

1. *the 1-eigenspace of ad_a is $\mathbb{F}a$.*

Definition (J. Hall, F. Rehren, S. Shpectorov 2015)

An axial algebra of Monster type (α, β) is a pair (A, X), where

- *A is a commutative algebra and*
- *X is a set of $\mathcal{F}$-axes that generates A.*

The algebra is called primitive if the elements of X are all primitive.

- Associative algebras with identity generated by idempotents are axial algebras of Monster type (α, β), for every α and β.
- Jordan algebras generated by idempotents are axial algebras of Monster type $(\frac{1}{2}, \beta)$, for every β.
- axial algebras of Jordan type η are axial algebras of Monster type both (η, β) and (α, η).
- the Griess algebra is an axial algebra of Monster type $(\frac{1}{4}, \frac{1}{32})$.

Lemma (M. Stout (2021))

Over a field $\mathbb{F}$ of characteristic 2, every axial algebra of Monster type (or Jordan type) is associative.

From now on we assume $\mathrm{char}(\mathbb{F}) \neq 2$.

$\mathbb{Z}_2$-grading and Miyamoto involutions

Let a be an axis in A. Define

$$A_+(a) := A_{\{0,1,\alpha\}}(a) \quad \text{and} \quad A_-(a) := A_\beta(a)$$

We get a grading of the algebra, that is

$$A = A_+(a) \oplus A_-(a)$$

and

$$A_+(a)A_+(a) \subseteq A_+(a)$$

$$A_+(a)A_-(a) \subseteq A_-(a)$$

$$A_-(a)A_-(a) \subseteq A_+(a)$$

In the context of VOA's, Miyamoto made the following observation

Lemma (Miyamoto 1996)

The map that negates $A_-(a)$ and induces the identity on $A_+(a)$ is an involutory algebra automorphism called Miyamoto involution.

The group generated by all the Miyamoto involutions associated to the axes in X is called the Miyamoto group $\mathrm{Miy}(X)$.

The Miyamoto group is not always the full automorphism group of the algebra.

In the case of the Griess algebra, the Miyamoto group is the Monster group.

In the study of axial algebras of Monster type it is natural to start with the 2-generated objects, that is algebras generated by two axes.

In the special case of Majorana algebras, the Norton-Sakuma theorem and its generalizations gives us the classification of such 2-generated algebras.

We would like to find a similar classification for axial algebras of Monster type (α, β).

Primitive 2-generated axial algebras of Monster type

From now on we assume that (A, X) is an axial algebra of Monster type (α, β), and $X = \{a_0, a_1\}$. We also assume that a_0 and a_1 are primitive.

Definition

If there exists an automorphism of A swapping a_0 and a_1, we say that A is symmetric. Otherwise, A is non-symmetric.

Axial algebras of Monster type (α, β), for arbitrary $\alpha, \beta \in \mathbb{F}$, was first studied by Rehren in his PhD thesis (2015). There are three different cases to consider:

- $\alpha \notin \{2\beta, 4\beta\}$
- $\alpha = 2\beta$
- $\alpha = 4\beta$

Rehren constructed several examples of 2-generated axial algebras of Monster type (α, β) generalising the Norton-Sakuma algebras (the subalgebras of the Griess algebra generated by two axes):

$$3A(\alpha,\beta),\ 4A(\tfrac{1}{4},\beta),\ 4B(\alpha,\tfrac{\alpha^2}{2}),\ 5A(\alpha,\tfrac{5\alpha-1}{8}),\ 6A(\alpha,\tfrac{-\alpha^2}{4(2\alpha-1)})$$

All these algebras are symmetric.

More new examples have been constructed, in the case $\alpha = 2\beta$, by Galt, Joshi, Mamontov, Staroletov and Shpectorov (2020), as subalgebras of Matsuo algebras (double axis construction):

$$4J(2\beta,\beta),\ 6J(2\beta,\beta),\ Q_2(2\beta,\beta)$$

Here, the first two algebras are symmetric, the last one is non-symmetric.

Dimension

Theorem (Rehren 2015, FMS 2019-2024)

If $(\alpha, \beta) \neq (2, \frac{1}{2})$, then A has dimension at most 8.

The bound 8 is attained.
The case $(2, \frac{1}{2})$ is a true exception.

Theorem (FMS 2019)

There exists a symmetric 2-generated axial algebra $\mathcal{H}$ of Monster type $(2, \frac{1}{2})$ of infinite dimension.

The symmetric case

In 2020, Yabe constructed independently the algebra $\mathcal{H}$ plus 5 more new symmetric algebras

$$4Y(\tfrac{1}{2},\beta),\ 4Y(\alpha,\tfrac{1-\alpha^2}{2}),\ 6Y(\tfrac{1}{2},2),\ IY_3(\alpha,\tfrac{1}{2},\mu),\ IY_5(\alpha,\tfrac{1}{2})$$

Furthermore, he obtained an almost complete classification of symmetric algebras.

Theorem (Yabe 2020)

Let A be a symmetric 2-generated axial algebra of Monster type (α,β) over a field $\mathbb{F}$. Then, one of the following occurs

1. *A is of Jordan type α or β*
2. *A is one of the 12 algebras listed above or a quotient of it*
3. *A is a quotient of $\mathcal{H}$*
4. $\operatorname{char}\mathbb{F}=5$ *and A has axial dimension greater than 5.*

One more algebra was missing in characteristic 5: the algebra $\hat{\mathcal{H}}$, which is a cover of $\mathcal{H}$.

Theorem (FM 2021)

Let A be a symmetric 2-generated axial algebra of Monster type (α, β) *over a field* $\mathbb{F}$ *of characteristic 5. Then one of the following occurs*

1. *A is of Jordan type* α *or* β*,*
2. *A is one of the algebras in Yabe's list,*
3. *A is a quotient of* $\hat{\mathcal{H}}$*.*

Quotients of $\mathcal{H}$ and $\hat{\mathcal{H}}$ has been classified (F., McInroy, Mainardis 2024):

- all quotients give symmetric algebras
- there is no bound on the dimensions of the quotients

Non symmetric algebras

First key observation: A contains two large symmetric subalgebras.

Let τ_0 and τ_1 be the Miyamoto involutions associated to a_0 and a_1, respectively. Set

$$a_2 := a_0^{\tau_1} \quad a_{-1} := a_1^{\tau_0},$$

and

$$A_e := \langle\langle a_0, a_2 \rangle\rangle, \;\; A_o := \langle\langle a_{-1}, a_1 \rangle\rangle$$

Lemma

A_e and A_o are symmetric 2-generated axial algebras.

Second key observation: A_e and A_o must “match" in the sense that one of the following holds

- $|a_0^{Miy(X)}| = |a_1^{Miy(X)}|$
- $|a_0^{Miy(X)}| = 2|a_1^{Miy(X)}|$
- $2|a_0^{Miy(X)}| = |a_1^{Miy(X)}|$

This gives us a strategy to deal with non-symmetric algebras.

Third key observation: if $\alpha \neq 4\beta$, every algebra is a quotient of a 8-dimensional universal algebra over a polynomial ring. This implies

- every algebra is determined, up to quotients, by 4 parameters which satisfy certain polynomial equations.
- we know some relations between the axes of the algebra

An example (M. Turner)

Let A be the Matsuo algebra $3C(\alpha)$, with $\alpha \neq -1, \frac{1}{2}$.
A has basis x, y, z and product defined by

$$ab := \frac{\alpha}{2}(a + b - c)$$

whenever $\{a, b, c\} = \{x, y, z\}$.
$(A, \{x, y\})$ is an axial algebra of Jordan type α.

Moreover, A has an identity element

$$\mathbb{1} := \frac{1}{\alpha+1}(x + y + z).$$

Set

$$a_0 := \mathbb{1} - x \quad \text{and } a_1 := y.$$

Then $(A, \{a_0, a_1\})$ is an axial algebra of Monster type $(\alpha, 1 - \alpha)$.

Case $\alpha = 2\beta$

Theorem (FMS 2021)

Let A be a 2-generated axial algebra of Monster type $(2\beta, \beta)$ over any field of characteristic other than 2. Then one the following occurs

1. *A is symmetric*
2. *A is isomorphic to $Q_2(\beta)$ or to its 3-dimensional quotient (when $\beta = -\frac{1}{2}$)*
3. *A is the algebra of Turner's example with $\alpha = \frac{1}{3}$.*

Recently, M. Turner classified the skew 2-generated axial algebras of Monster type (α, β), that is algebras such that $|a_0^{Miy(X)}| \neq |a_1^{Miy(X)}|$.

Theorem (Turner 2024)

Let A be a 2-generated primitive skew axial algebra of Monster type (α, β) over a field $\mathbb{F}$. Then we have one of the following:

1. *A is the is the algebra of Turner's example with $\alpha + \beta = 1$ and $\alpha \neq \frac{1}{2}$;*
2. *$\mathbb{F}$ has characteristic other than 5 and A is $Q_2(\frac{1}{3}, \frac{2}{3})$;*
3. *$\mathbb{F}$ has characteristic 5 and A is $Q_2(\frac{1}{3})^{\times} \oplus \langle \mathbb{1} \rangle$.*

We are working on the remaining cases.

Thank you!