On the Packing Density of Lee Spheres

Yue Zhou, yue.zhou.ovgu@outlook.de

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Outline

- Introduction
- The “hugging” number of Lee spheres
- Lower bounds for the lattice packing density
- Upper bounds for the lattice packing density
- Concluding Remarks

1. Introduction

Introduction

- For any $x = (x_1, \cdots, x_n),\ y = (y_1, \cdots, y_n) \in \mathbb{Z}^n$, the **Lee distance** ($\ell_1$-norm, Manhatten distance...) between them is $d_L(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.
- Lee sphere of radius r centered at O is:

$$\boldsymbol{S(n,r)} := \{(x_1, \cdots, x_n) \in \mathbb{Z}^n \colon \sum_{i=1}^{n} |x_i| \leq r\}$$

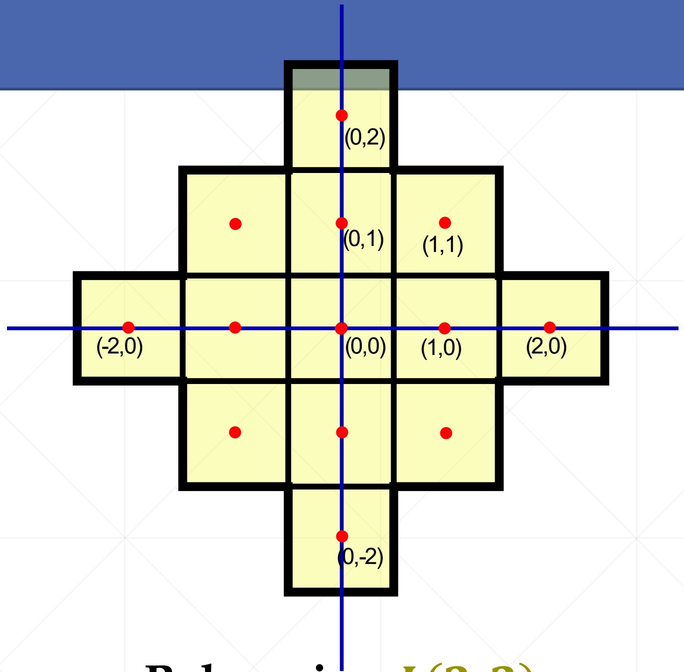
$$|S(n,r)| = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{n}{i} \binom{r}{i}.$$

- A **perfect Lee code** C $\Leftrightarrow$ A **tiling** of $\mathbb{Z}^n$ by translates of $S(n,r)$

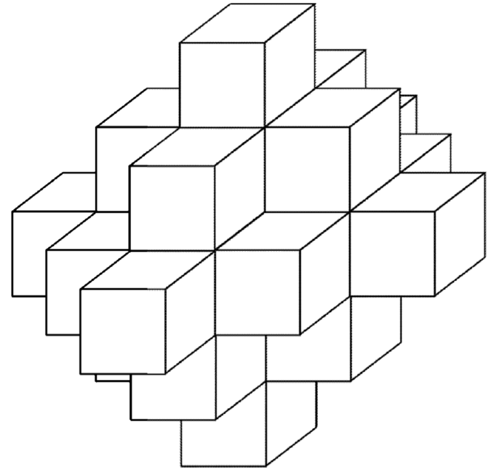
$$\mathbb{Z}^n = \dot{\bigcup}_{c \in C} (S(n,r) + c) = S(n,r) \oplus C$$

- It is equivalent to ``tile" $\mathbb{R}^n$ by $\boldsymbol{L(n,r)} = \boldsymbol{S(n,r)} + \left[-\frac{1}{2}, \frac{1}{2}\right]^n$

$$\mathbb{R}^n =_{a.e.} \mathbb{Z}^n + \left[-\frac{1}{2}, \frac{1}{2}\right]^n =_{a.e.} \boldsymbol{L(n,r)} \oplus C$$



Polyomino $L(2,2)$ associated with $S(2,2)$

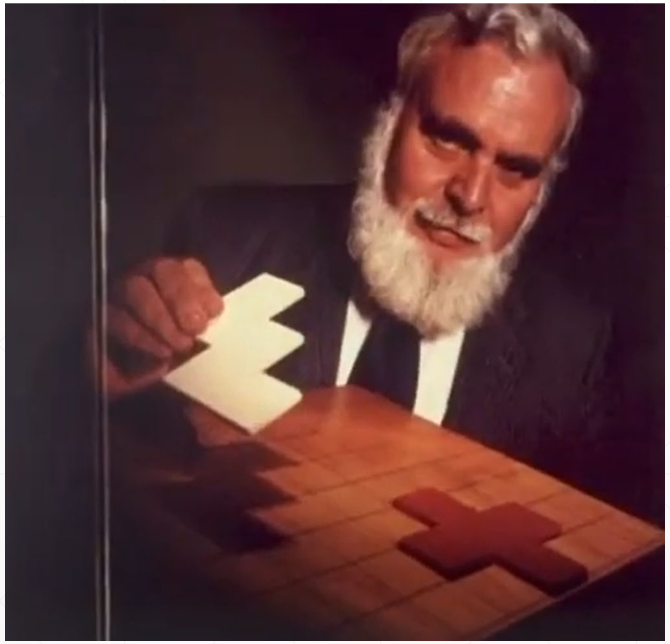


Polyomino $L(3,2)$

Introduction

- **Theorem (Golomb, Welch 1968/1970)** Perfect Lee codes exist for $n = 1,2$ and any r; and for $r = 1$ and any n.

- **Golomb-Welch conjecture** (1968): there are no more perfect Lee codes for other choices of n and r.



Solomon Golomb (1932-2016)

5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4
8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7
11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2
6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5
9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8
12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3
7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6
10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1

$n = 2, r = 2$, the green points form a perfect code (lattice)

Introduction

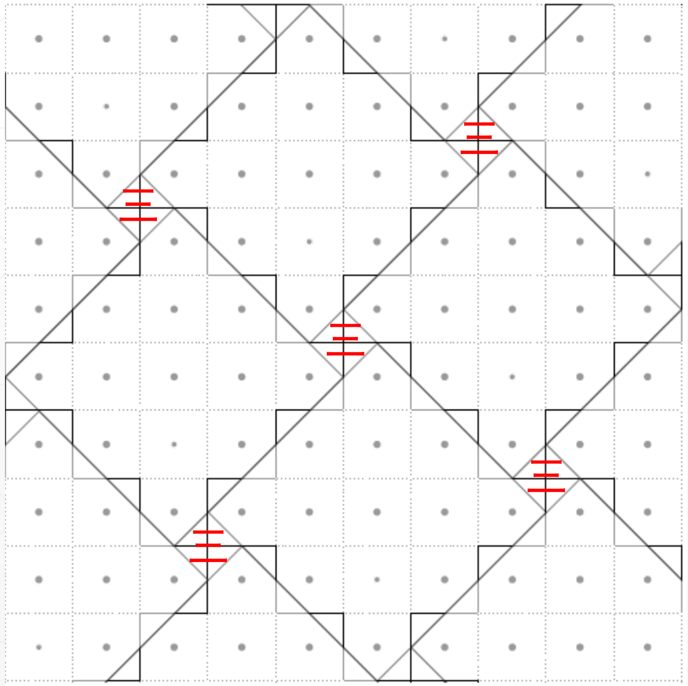
- **GW conjecture**: $\nexists$ Perfect Lee codes for $n \geq 3$ and $r \geq 2$.
- partially proved for given n and $r > N(n)$.

Basic Idea by GW

- Conscribed cross-polytope $X(n,r)$ of $L(n,r)$,

$$\mathrm{vol}\big(X(n,r)\big) = \frac{(2r+1)^n}{n!}$$

$$\mathrm{vol}\big(L(n,r)\big) = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{r}{i}\binom{n}{i} \approx 2^n \binom{r}{n}, r \to \infty.$$



- The packing density of $X(n,r)$ must be smaller than $(0.87)^n$, n large enough. (Tóth, Fodor, Vígh, 2015)

Introduction

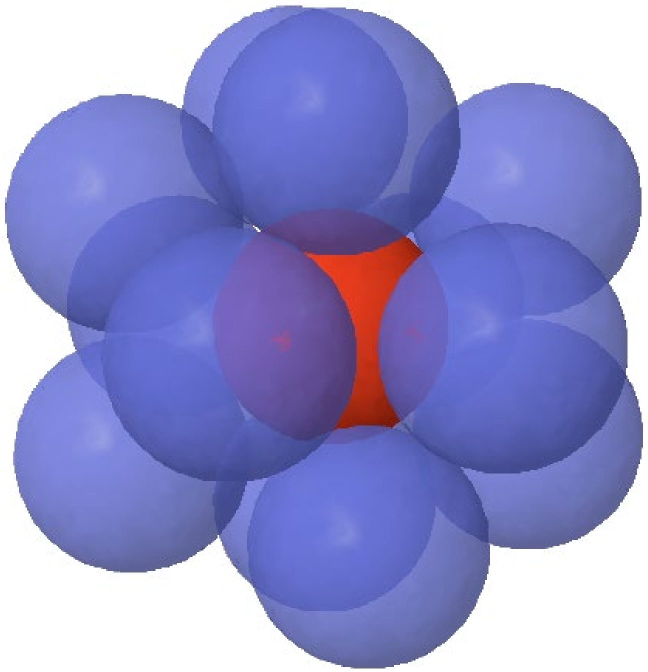
- GW conjecture was partially proved for given n and $r > N(n)$.
- Many works to find small $N(n)$. (Combinatorial, Geometric and Functional Analysis)
  - Post(1975), Astola, Lepistö (1980's), Horak, Kim (2018), etc.
  - Best results up to now:

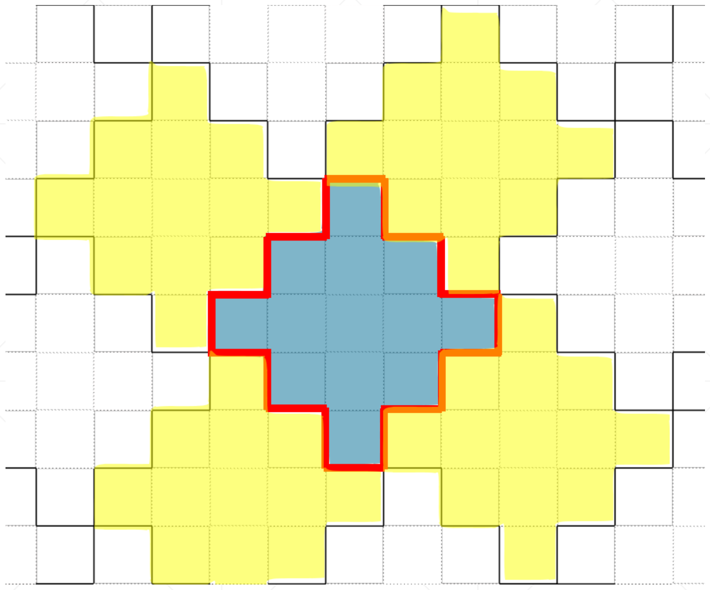
$$N(n) \leq \sqrt{2}\sqrt{n+c} \text{ if } n \text{ is large enough and } c \text{ is a small constant}$$

  - Survey: Horak, Kim, 50 years of the Golomb-Welch conjecture, 2018.
- Lee codes in $\mathbb{Z}_q^n$, and weak GW-conjecture
  - Association scheme (Sole 1990's), weak-metric=multivariate polynomial, Lloyd Theorem…
  - linear programming method, SDP method (Astola, Polak, 2015-2019).

The ``hugging” number of Lee spheres

The hugging number of Lee spheres





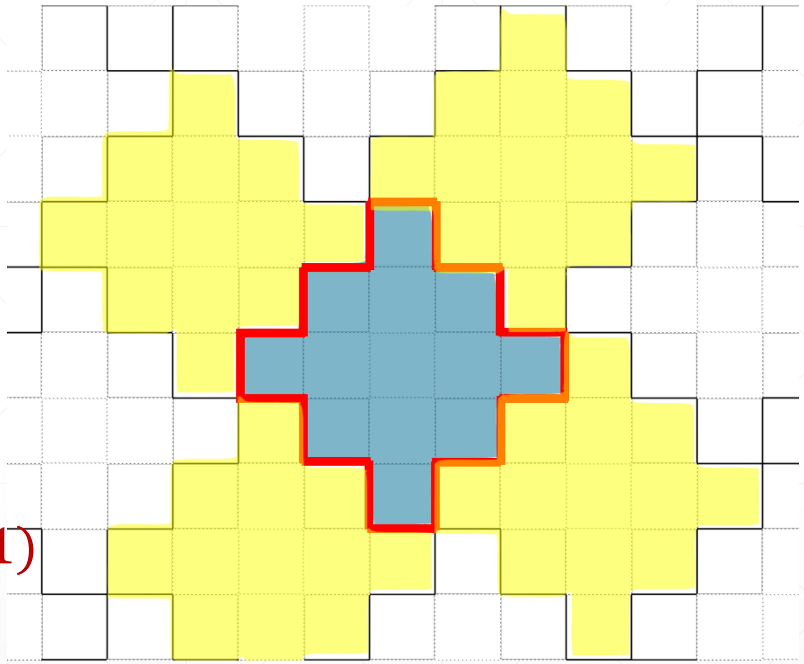
- The **kissing number** is the greatest number of non-overlapping **unit spheres** that can be arranged such that they each touch a **common unit sphere**.
- If there exists a tiling or a very dense packing of Lee spheres, there should be a large **hugging number** which is the greatest **number of facets** of $\boldsymbol{S}(\boldsymbol{n}, \boldsymbol{r})$ that can be covered by non-overlapping translations of $\boldsymbol{S}(\boldsymbol{n}, \boldsymbol{r})$.

The hugging number of Lee spheres

Proposition (Z. 2024+). There are asymptotically perfect ``hugging" of $S(n,r)$ by its translation, for fixed r and $n \to \infty$.

Proof (sketch). When n is large enough, we only have to care about elements of weight r in $S(n,r)$ in the shape of

$$(\overbrace{\pm 1, \pm 1, \cdots, \pm 1}^{r}, 0, 0, \cdots).$$



- Number of facets of $S(n,r)$: $2^r \binom{n}{r} (2n - r) + O(n^r)$
- A translation $S(n,r) + (\underbrace{1,1,\cdots,1}_{2r+1},0,0,\cdots)$ can cover $\binom{2r+1}{r}(r+1)$ facets
- By Liu, Shangguan 2024+, there exists a constant $(2r+1)$-weight code $\mathcal{C}$ in $\mathbb{F}_3$ of minimum distance $2r+1$ with $\#\mathcal{C} \geq \frac{(1-o(1))2^{r+1}\binom{n}{r+1}}{\binom{2r+1}{r+1}}$.
- $\binom{2r+1}{r}(r+1) \cdot \frac{(1-o(1))2^{r+1}\binom{n}{r+1}}{\binom{2r+1}{r+1}} = (1-o(1))2^{r+1}\binom{n}{r}(n-r) \approx 2^r \binom{n}{r}(2n-r) + O(n^r),\ n \to \infty$

The hugging number of Lee spheres

- Recall that the packing density of cross-polytopes must be $\ll (0.87)^n$, n large enough.
- Geometric approach: GW conjecture was partially proved for given n and $r > N(n)$.
- **Question**: How about the cases with fixed r and large n?
- Our ``hugging number“ results show that it could be locally very dense and the geometric intuition does not always work.
- Next we show the packing density of $S(n, r)$ with $n \to \infty$ is $>$ some constant.

The density of (lattice) packings

- $\delta_T(S(n,r))$, $\delta_L(S(n,r))$: the supremum of the upper density of translative/lattice packings of translates of $S(n,r)$.
- If $\delta_T(S(n,r)) < 1$ for $n \geq 3$ and $r \geq 2$, then GW-conjecture is proved; the converse seems not true.
- If $\delta_L(S(n,r)) < 1$ for $n \geq 3$ and $r \geq 2$, then the lattice version of GW-conjecture is proved; the converse is true.

Lower bounds for the lattice packing density $\delta_L(S(n,r))$

Lower bounds

A typical algebraic packing theorem.

Theorem 1. Let G be a finite abelian group, $R = \{a_1, a_2, \cdots, a_n\} \subseteq G$. Assume that R generates G. Define a homomorphism $\varphi: \mathbb{Z}^n \to G$ by $\varphi(e_i) = a_i$, where $e_i = (0, \dots, 0,1,0, \dots)$. Then the following statements are equivalent:

i-th

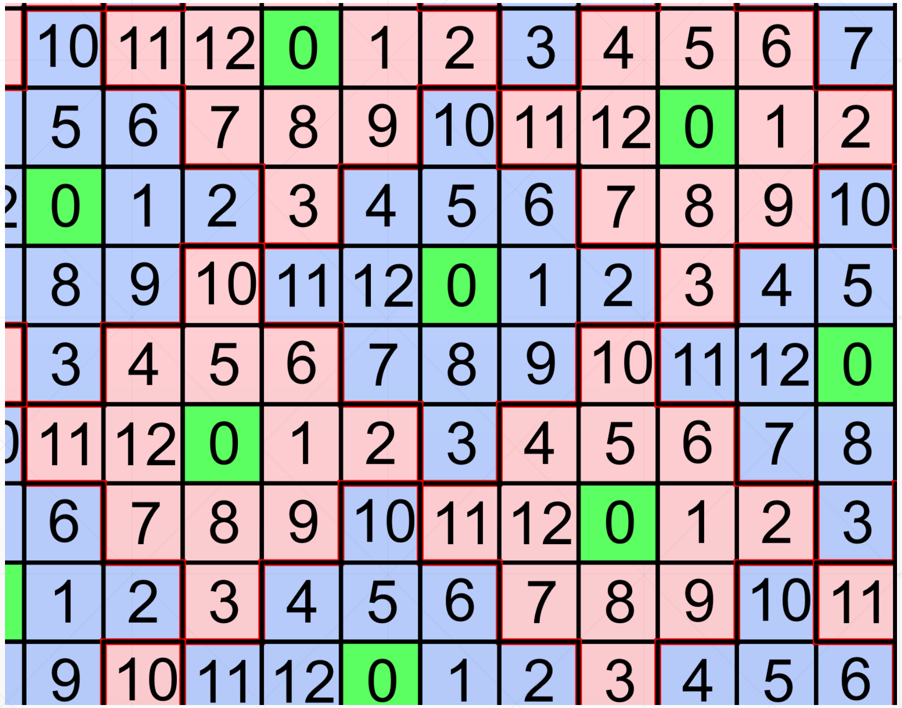
a) The restriction of φ on $S(n,r)$ is injective.

b) The set $\ker(\varphi)$ defines a lattice packing of $S(n,r)$.

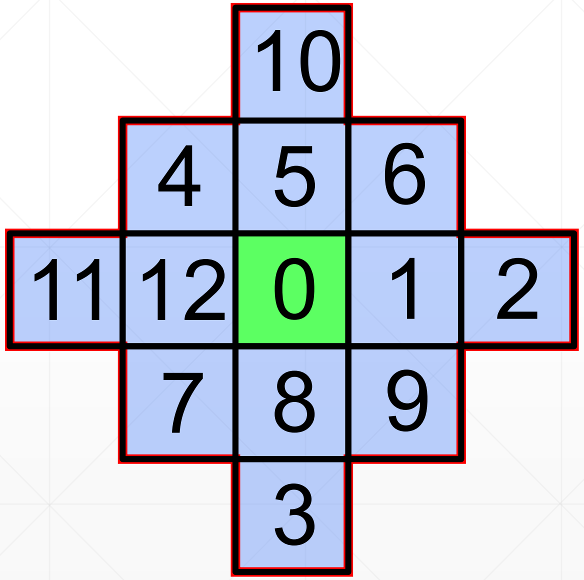
The density of the lattice packing is $\frac{|S(n,r)|}{|G|}$.

Lower bounds

Theorem 1 *(Horak, AlBdaiwi 2012) $\exists$ a lattice tiling of $\mathbb{Z}^n$ by Lee spheres of radius $r \Leftrightarrow$ there are an abelian group G of order $|S(n,r)|$ and a homomorphism $\varphi: \mathbb{Z}^n \mapsto G$ such that $\varphi|_{S(n,r)}$ is a bijection.



$n = 2, r = 2$



$G = (\mathbb{Z}_{13}, +), \varphi(e_1) = 1, \varphi(e_2) = 5$

Lower bounds

Theorem 1. Let G be a finite abelian group, $R = \{a_1, a_2, \cdots, a_n\} \subseteq G$. Assume that R generates G. Define homomorphism $\varphi: \mathbb{Z}^n \to G$ by $\varphi(e_i) = a_i$. Then the following statements are equivalent:

a) The restriction of φ on $S(n, r)$ is injective.

b) The set $\ker(\varphi)$ defines a lattice packing of $S(n, r)$.

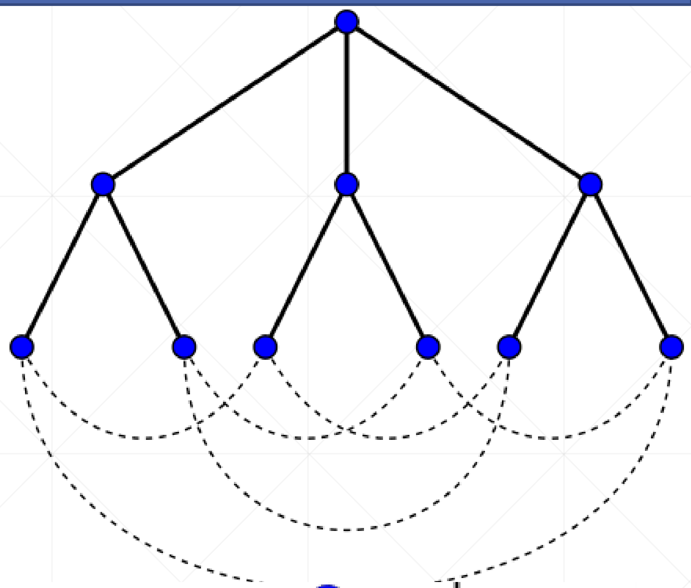
The density of the lattice packing is $\frac{|S(n,r)|}{|G|}$.

Goal: Find a finite (additive) abelian group G, and $R \subseteq G$ such that

(1) R generates G;

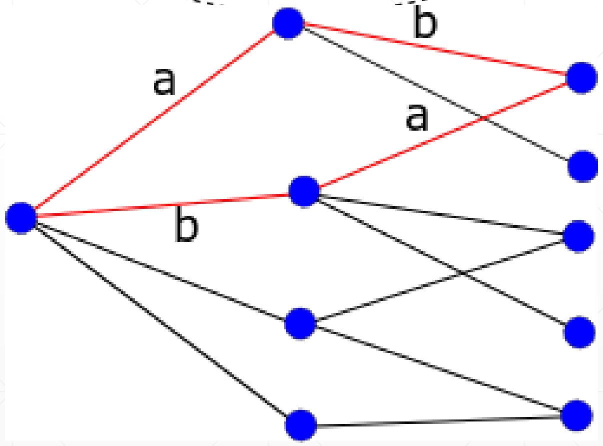
(2) For any $u, v \in \mathbb{Z}^n$ with $\|u\|_1, \|v\|_1 \leq r$, if $\sum u_i a_i = \sum v_i a_i$, then $u = v$.

A related problem in Graph Theory



Degree/diameter problems

- **Moore bound** for general graphs: $\#V \leq 1 + d\sum_{i=0}^{k-1}(d-1)^i$.
- **Moore-like bound** for abelian Cayley graphs:



$$|G| \leq \sum_{i=0}^{\min\{k,d\}} 2^i \binom{k}{i}\binom{d}{i}.$$

- Moore-like bound = $|S(n,r)|$ with $n = d, r = k$.
- Abelian Cayley graph meeting the Moore-like bound⇔Lattice tiling of $S(n,r)$

Lower bounds

Theorem 2. For any $r > 1$, choose $\mathbb{F}_q$ such that $\mathrm{char}(\mathbb{F}_q) > r + 1$. Let $G = C_{2r+1} \times \mathbb{F}_q^r$, and $R = \{(1, x, x^2, \cdots, x^r): x \in \mathbb{F}_q^*\}$. Then R satisfies (1) and (2) with $n = q - 1$.

(1) R generates G;

(2) For any $u, v \in \mathbb{Z}^n$ with $\|u\|_1, \|v\|_1 \leq r$, if $\sum u_i a_i = \sum v_i a_i$, then $u = v$.

Density:

$$\frac{|S(n,r)|}{(2r+1)(n+1)^r} \to \frac{2^r}{(2r+1)r!}, \quad n \to \infty$$

- In fact, R is a Sidon set of order r: the sum $\sum_{j=1}^{r} b_{i_j}, 1 \leq i_1 \leq \cdots \leq i_r \leq q - 1$ are all distinct. One can always use this trick to construct a lattice packing of Lee spheres. (Kovačević 2022)

Lower bounds

For $r = 2$, the density of the previous construction tends to $2/5$.

A better construction for $r = 2$:

Theorem 3. (Xiao, Z. 2024) Let q be an odd prime power $q \equiv 2 \pmod 3$. Define $R = \{(1, x, x^2): x \in \mathbb{F}_q^*\} \subseteq C_3 \times \mathbb{F}_q^2$. Then R satisfies (1) and (2) with $n = q - 1$.

The packing density is $\frac{2n^2+2n+1}{3(n+1)^2} \to \frac{2}{3}$, $n \to \infty$.

- Extra work to show $\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 = 0 \end{cases}$ has no solutions in $\mathbb{F}_q^*$.
- Similar construction can be done for q even.
- Related to planar functions in finite geometry.

Upper bounds for the lattice packing density $\delta_L(S(n,r))$

Upper bounds

Recall that the **geometric method** can only handle fixed n and $r > N(n)$.

For fixed r, **Algebraic and Combinatorics Methods:**

- Symmetric polynomials over finite fields (Kim 2017, Zhang, Ge 2017, Qureshi 2020)
  - Fast algorithm for small n;
  - Works for infinitely many n?
  - Usually, $|S(n,r)|$ needs to be prime or to have large prime divisors.
- Convert the original problem into a group ring equation
  - Group characters (=eigenvalue of the associated graph), algebraic number theory, finite fields…(Zhang, Z. 2019)
    - Usually need small prime divisors of $|S(n,r)|$.
  - Handle the group ring equations directly mod 3, mod 5… (Leung, Z. 2020)
    - Currently only works for $r = 2$.

Group Ring Equations approach

A lattice tiling of Lee spheres of radius 2 in $\mathbb{Z}^n$ $\Leftrightarrow$ The existence of $T \subseteq G$, where G is an abelian (multiplicative) group of order $2n^2+2n+1$, such that $T = T^{(-1)}$, the identity $e \in T$ and

$$T^2 = 2G - T^{(2)} + 2ne \in \mathbb{Z}[G],$$

where $T^{(s)} := \sum_{t \in T} t^s$.

[Zhang, Z. 2019] Apply $\chi \in \hat{G}$, obtain equations in algebraic integer rings.

[Leung, Z. 2020] Analyze $T^3 \equiv T^{(3)} \bmod 3, T^5 \equiv T^{(5)} \bmod 5$

- **Theorem** (Leung, Z. 2020) For $n \geq 3$ and $r = 2$, lattice tiling of $\mathbb{Z}^n$ by $S(n,2)$ does not exist, i.e. $\delta_L(S(n,r)) < 1$.
- **Theorem** (with Xu 2023, Z.J. Zhou 2024, Xiao 2024+) For $n \geq 3$ and $r = 2$, almost perfect lattice packing of $\mathbb{Z}^n$ by $S(n,2)$ does not exist, i.e. $\delta_L(S(n,r)) < \frac{\#S(n,2)}{\#S(n,2)+1}$.

Symmetric polynomial approach to exclude small n

The difficulty of group ring equations

- $r = 2$: Perfect case (15 pages) → Almost perfect case (50+ pages)
- Group ring becomes more complicated for large r:
  - $r = 2$: $T^2 = 2G - T^{(2)} + 2ne$;
  - $r = 3$: $T^3 = 6G - 3T^{(2)}T - 2T^{(3)} + 6nT$;
  - $r = 4$: $T^4 = 24G - 12n\left(T^2 + T^{(2)}\right) - 6T^{(2)}T^2 - 3T^{(2)}T^{(2)} - 8\,T^{(3)}T - 6T^{(4)} + 12n(n-1)$;
  - $r = 5$: … …

Symmetric polynomial approach

Theorem 1 * The following three conditions are equivalent:

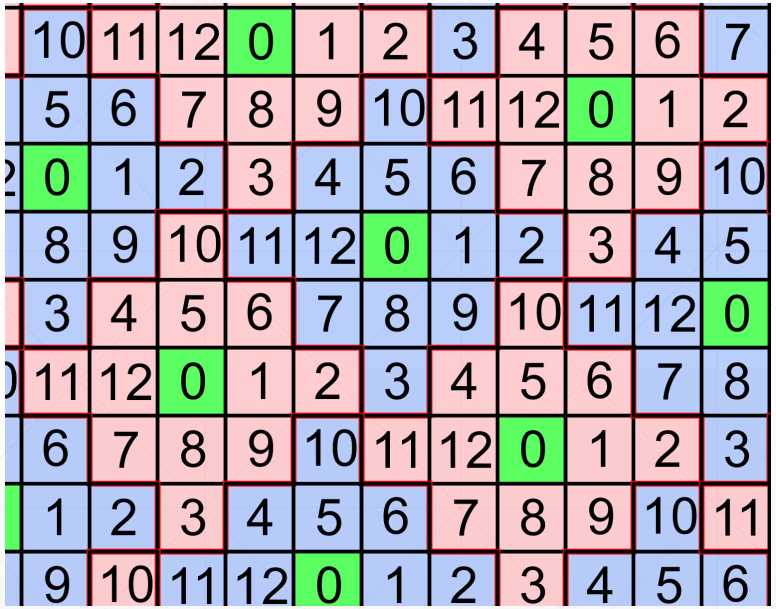
a) $\exists$ a lattice tiling of $\mathbb{Z}^n$ by Lee spheres of radius r

b) there are an abelian group G of order $|S(n,r)|$ and a homomorphism $\varphi: \mathbb{Z}^n \mapsto G$ such that $\varphi|_{S(n,r)}$ is a bijection

c) $\exists$ abelian (additive) group G of order $|S(n,r)|$ and $\exists R = \{x_1, \dots, x_n\} \subseteq G$, such that $\left\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^n, \|u\|_1 \leq r\right\} = G$.

Example: $R = \{1,5\} \subseteq G = C_{13}$.

$$\left\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^2, \|u\|_1 \leq 2\right\} = \{0, \pm 1, \pm 5, \pm 2, \pm 10, \pm 1 \pm 5\} = C_{13}$$



Upper bounds

$$\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^n, \|u\|_1 \le r\} = G \text{ with } |G| = |S(n,r)| = \sum_{i=0}^{\min(n,r)} 2^i \binom{n}{i}\binom{r}{i}.$$

The idea by Kim ($r = 2$), generalized by Zhang & Ge, and Qureshi ($r \ge 2$):

- Suppose that $|G| = pm$, define projection $\varphi: G \to (\mathbb{F}_p, +), \bar{x} := \varphi(x)$. Consider

$$Q_{(n,r)}^k(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{u \in \mathbb{Z}^n : \|u\|_1 \le r} \left(\varphi\left(\sum_{x_i \in R} u_i x_i\right)\right)^{2k} = \sum_{u \in \mathbb{Z}^n : \|u\|_1 \le r} \left(\sum_{x_i \in R} u_i \bar{x}_i\right)^{2k} = \sum_{g \in G} \varphi(g)^{2k} = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

- By expanding,

$$Q_{(n,r)}^k(\bar{x}_1, \cdots, \bar{x}_n) = \sum_\lambda c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \ne (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n), \qquad \text{------(\#)}$$

where $S_\lambda = S_{\lambda_1} \cdots S_{\lambda_\ell}$ and $(\lambda_1, \cdots, \lambda_\ell)$ is a partition of $2k$ with $\ell \le r$ and $S_m(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{i=1}^n \bar{x}_i^m$.

- [Kim 2017, for $r = 2$] If $m = 1$ and we can determine all $S_2, S_4, \ldots, S_{2n} = 0$ then $e_n = x_1^2 \cdots x_n^2 = 0$ (by Newton's identity), where e_i is the elementary symmetric polynomial of $x_1^2, \cdots, x_n^2$, then a contradiction!

- [Qureshi 2020] If $p \nmid m$ and the leading coefficients $c_{(2k)} \ne 0$ in $\mathbb{F}_p$ for $k = 1, \cdots, \frac{p-1}{2}$ and $c_{(p-1)} = 0$, then $S_2 = S_4 = \cdots = S_{p-3} = 0$ which implies $c_{(p-1)} S_{p-1} = -m \ne 0$, a contradiction!

Example for $r = 3$

$$Q^k_{(n,3)}(\bar{x}_1, \cdots, \bar{x}_n)$$

$$= \left(\frac{2 \times 9^k}{3} + (2n+1)4^k + 4n^2 + 4n + 2\right) S_{2k} + \sum_{t=1}^{k-1} (4^t + 4^{k-t} + 4n + 2) \frac{(2k)!}{(2t)!\,(2k-2t)!} S_{2t} S_{2k-2t}$$

$$+ \frac{4}{3} \sum_{i=1}^{k-1} \sum_{j=1}^{i-1} \frac{(2k)!}{(2j)!\,(2i-2j)!\,(2k-2i)!} S_{2j} S_{2i-2j} S_{2k-2i} = \begin{cases} 0, & v-1 \nmid 2k; \\ -m, & v-1 \mid 2k. \end{cases}$$

- For $3 \le n \le 100$, Qureshi's approach can only exclude $n = 6,12,21,39,48,64,66,75,93$.

9 integers

Upper bounds

$$c_{(2k)}S_{2k}(\bar{x}_1,\cdots,\bar{x}_n)+\sum_{\lambda\neq(2k)}c_\lambda S_\lambda(\bar{x}_1,\cdots,\bar{x}_n)=\begin{cases}0, & p-1\nmid 2k;\\ -m, & p-1\mid 2k.\end{cases}\quad\text{---(\#)}$$

Use (#) to derive the following symmetric polynomials:

1. power sum polynomials sequence $S_2, S_4, \ldots, S_{2k}, \ldots$, with indeterminants $X_1, X_2, \ldots,$
2. Use S_i and Newton's identity

$$ke_k=\sum_{i=1}^{k}(-1)^{i-1}e_{k-i}p_i,$$

to determine elementary symmetric polynomials sequence $e_1, e_2, \ldots, e_n, \ldots,$ with indeterminants $Y_1, Y_2, \ldots,$ where $e_k=\sum\bar{x}_{i_1}^2\cdots\bar{x}_{i_k}^2$ and $p_i=S_{2i}$.

- There are several cases leading to contradictions.

Some necessary conditions

$$[S_2, S_4, \dots, S_{p-1}, \dots], \; S_{2k} = \sum_{i=1}^{n} \bar{x}_i^{2k},$$

$$[e_1, e_2, \dots, e_n, \dots], \; e_k = \sum \bar{x}_{i_1} \bar{x}_{i_2} \cdots \bar{x}_{i_k}.$$

- $[S_{2i}]_{i=1,2,\dots}$ must be of period $\frac{p-1}{2}$;
- $S_{p-1} \equiv$ the number of nonzero elements in $\bar{x}_i's \pmod{p}$;
- $e_{n+1} = e_{n+2} = \cdots = 0$;
- There should not be too many zero's in $\bar{x}_i's$.

Case I: $S_{p-1} > n$

Example 1. For $r = 3, n = 26, |G| = 24857 = 7 \times 53 \times 67, p = 67, m = 371.$

$$S = [0, \dots, 0, S_{2\times15} = X, 0, \dots, 0, S_{2\times26} = 0, \dots S_{2\times33} = 33, \dots]$$

$$S_{66} = \sum_{i=1}^{n} \bar{x}_i^{67-1} = 33 > 26$$

Case II: $e_j \neq 0$ for $j > n$

Example 2. $n = 40$, $|G| = 88641 = 3^3 \times 7^2 \times 67, p = 67, m = 1323$. Now

$$S = [0, \ldots, 0, S_{2\times 14} = X_1, 0, \ldots, 0, S_{2\times 18} = X_2, 0, \ldots 0, S_{2\times 33} = 33, \ldots]$$

and

$$e = [0,0,0,0,0,0,0,0,0,0,0,0,0,43X_1, 0,0,0,26\,X_2, 0,0,0,0,0,0,0,0,0,47X_1^2, 0,0,0, 21X_1X_2, 1,0,0,64X_2^2, 0,0,0,0,0,41X_1^3, 0,0,0,51X_1^2X_2, 34\,X_1, 0,0,62X_1X_2^2, 20X_2, 0,0,26X_2^3, 0,45X_1^4, 0,0,0,6X_1^3X_2, 59X_1^2, 0,0,7X_1^2X_2^2, 19X_1X_2, 66, \ldots]$$

$e_{66} = 66 \neq 0.$

Case III: too many $e_i = 0$ for $i \le n$

Example 3. For $r = 3, n = 12, |G| = 2625 = 7 \times 5^3 \times 3, p = 7, m = 375.$

$$S = [0,0,1,0,0,1,0,0,1,0,0,1 \ldots]$$

$$e = [0,0,5,0,0,6,Y_1,0,0,2Y_1,0,0,Y_1,Y_2,0,0,2Y_2,0,0,],\ e_{13} = Y_1,\ e_{14} = Y_2$$

Recall $e_k = \sum \bar{x}_{i_1}^2 \cdots \bar{x}_{i_k}^2, \varphi: G \to (\mathbb{F}_p, +), \bar{x} := \varphi(x)$. Hence there are exactly $M = 6$ nonzero $\bar{x}_i$'s , i.e. $(n - M)$ x_i's belong to the subgroup $C_{375} = \ker \varphi \le G$

$$1 + 6(n - M) + 12\binom{n-M}{2} + 8\binom{n-M}{3} = 377 > m = 375$$

0

$\pm x_i, \pm 2x_i, \pm 3x_i$

$\pm x_i \pm x_j,$
$\pm 2x_i \pm x_j,$
$\pm x_i \pm 2x_j,$

$\pm x_i \pm x_j \pm x_k,$

Upper bounds

Our improvement (Xiao, Z. 2024+) for the nonexistence of lattice tiling of $\mathbb{Z}^n$ by Lee spheres $S(n,r)$ (i.e., $\delta_L\big(S(n,r)\big) < 1$):

- When $r = 3$, we can exclude every $3 \leq n \leq 1000$, except for $n = 122$ and 634.
- When each prime divisor of $|S(n,3)|$ is much smaller than n, it becomes difficult to get a contradiction. For instance, $n = 122$, $|S(n,3)| = 3 \times 5^2 \times 7^2 \times 23 \times 29$.
- A recursive formula for $c_\lambda(k)$ and for any r and large n in

$$c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \neq (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases} \quad \text{---}(\#)$$

- This approach also works for other r and lattice packings with density ≈ 1 (for instance, the almost perfect case).
- Projection from G to $\mathbb{Z}_{p_1^{i_1}} \times \cdots \times \mathbb{Z}_{p_s^{i_s}}$ (instead of $\mathbb{F}_p$) is also possible.

Concluding Remarks

- For fixed r, $\delta_L\big(S(n,r)\big) \to \frac{2^r}{(2r+1)r!}$, $n \to \infty$. In particular, $\delta_L\big(S(n,2)\big) \to \frac{2}{3}$, $n \to \infty$
- Almost perfect $\sim \delta_L = \frac{\#S(n,2)}{\#S(n,2)+1}$ only exists for $n = 1,2$.
- Symmetric polynomial method: Improved algorithm for radius > 2 and many small n.

Questions:

- How to prove $\delta_L\big(S(n,r)\big) < 1$ for infinitely many n?

$$\left(\frac{2\times 9^k}{3} + (2n+1)4^k + 4n^2 + 4n + 2\right) S_{2k} + \sum_{t=1}^{k-1}\left(4^t + 4^{k-t} + 4n + 2\right)\frac{(2k)!}{(2t)!(2k-2t)!} S_{2t}S_{2k-2t} +$$

$$\frac{4}{3}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\frac{(2k)!}{(2j)!(2i-2j)!(2k-2i)!} S_{2j}S_{2i-2j}S_{2k-2i} = \begin{cases} 0, & v-1 \nmid 2k; \\ -m, & v-1 \mid 2k. \end{cases} \quad (r=3)$$

$$|S(n,3)| = \frac{4}{3}n^3 - 2n^2 + \frac{8}{3}n + 1.$$

- General (non-lattice) cases?

Thanks for your attention!