On Yamada polynomials of spatial graphs and polynomials of related knots

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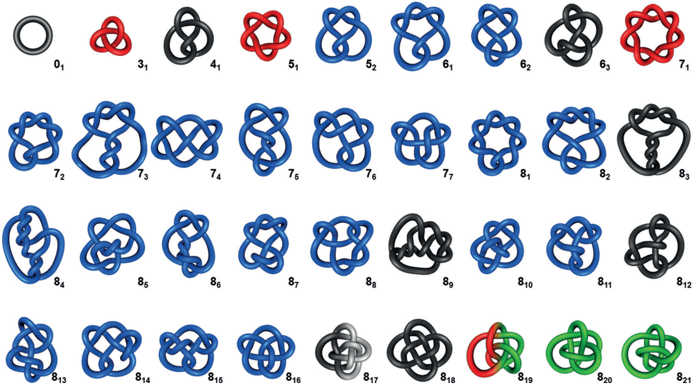
The talk is based on the join work with Olga Oshmarina (Novosibirsk State University):

O. Oshmarina, A. Vesnin, *Polynomials of complete spatial graphs and Jones polynomial of related links*. arXiv:2404:12264, 28 pp.

Knots and their diagrams

Knot is an embedding of S^1 in $\mathbb{S}^3$. Two knots K and K' are equivalent if there is an ambient isotopy h_t, $t \in [0,1]$, of $\mathbb{S}^3$ such that $h_0(K) = K$ and $h_1(K) = K'$.



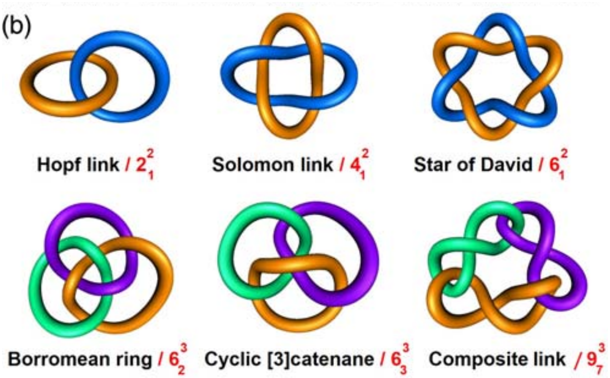
Enumeration of knots by number of crossings was started by Tait in1890. There are more than 350 millions of knots with at most 19 crossings.

[Picture from S.D.P. Fielden, D.A. Leigh, S.L. Woltering, Molecular knots. 2017]

[B. Burton, The Next 350 Million Knots, 2022.]

Links and their diagrams

Link is an embedding of a finite number of disjoint copies of S^1 in S^3.



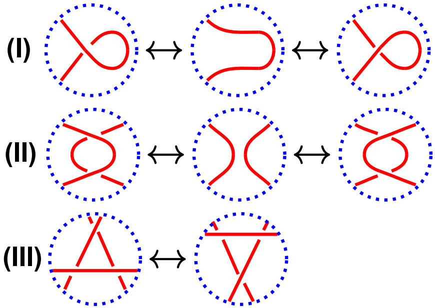
Links admitting diagrams with small number of crossings.

Two links are said to be equivalent if they are ambient isotopic.

[Picture form G.-H. Guo, Y. Jiao, Y. Feng, J.F.Stoddart, The Rise and Promise of Molecular Nanotopology. July 2021]

Theorem (K. Reidemeister, 1927)

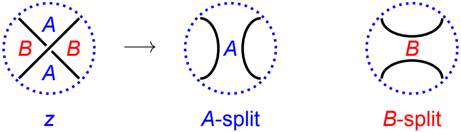
Two links K and K' in $\mathbb{S}^3$ are ambient isotopic if and only if a diagram of K can be transformed into a diagram of K' by a finite sequence of moves among (I)–(III) and plane isotopy.



States of a knot diagram

Let K be a link. Let D be a diagram of K. Assume that D has n crossings.

For any crossing $z \in D$ define local transformations of D: A-split and B-split:



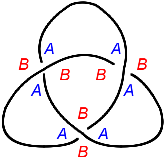
After splitting in all crossings of D we get a state S that is a collection of closed curves on a plane.

For any state S let $a(S)$ be number of A-splits, $b(S)$ number of B-splits, and $|S|$ number of connected components of S.

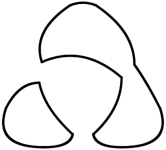
Observe that $a(S) + b(S) = n$ and diagram D has 2^n states.

Such (A, B)-labeling in a crossing point was used by Gauss's student Johann Benedict Listing in his book «Preliminary studies on topology», 1847.

Consider (A, B)-labelling for a diagram D_T of the trefoil knot T:



Labelling of D_T.



A state S of D_T.

Define a «bracket» polynomial

$$\langle D_K \rangle = \sum_S A^{a(S)} B^{b(S)} d^{|S|},$$

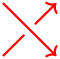
where the sum is taken over all states of a diagram D_K of a knot K.
For D_T we have 2^3 states and

$$\langle D_T \rangle = A^3 d^2 + 3A^2 B d + 3AB^2 d^2 + B^3 d^3.$$

Jones polynomial of a link

To make $\langle D_K \rangle$ invariant under Reidemeister moves suppose $B = A^{-1}$ and $d = -(A^2 + A^{-2})$. Then $\langle D_K \rangle$ is called the Kauffman bracket polynomial.

Give an orientation to link K and define writhe number of D_K by $w(D_K) = \sum_c \varepsilon(c)$ where sum is taken over all crossings.


$\varepsilon(c) = +1$


$\varepsilon(c) = -1$

Theorem (L. Kauffman)

Let D_K be a diagram of oriented link $K \subset \mathbb{S}^3$. Then Laurent polynomial

$$V(D_K; A) = (-A^3)^{-w(D_K)} \frac{\langle D_K \rangle}{(-A^2 - A^{-2})}$$

is an invariant of a link K under an ambient isotopy.

Denote $t = A^{-4}$. Then $V_K(t) = V(K; t^{-1/4}) \in \mathbb{Z}[t^{\pm 1/2}]$ is the Jones polynomial.

Vaughan Jones was awarded the Fields medal in 1990.

Spatial graphs (knotted graphs)

Let G be a graph with finite set V of vertices and finite set E of edges. Loops and multi-edges are admitted in G.

An embedding $f : G \to \mathbb{S}^3$ is called a spatial embedding of G, and $\mathcal{G} = f(G)$ is called a spatial G-graph.

If γ is a cycle in G then its spatial embedding $f(\gamma)$ is a knot in $\mathbb{S}^3$.

If $\lambda = \alpha \cup \beta$ is a couple of disjoint cycles in G then its spatial embedding $f(\lambda)$ is a 2-component link in $\mathbb{S}^3$.

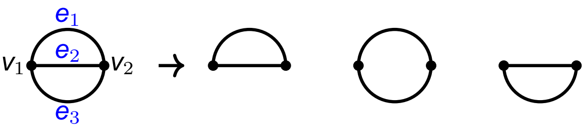
Thus, the theory of spatial graphs is a natural extension of the knot theory.

We will work in the piecewise-linear category and graphs are considered to be 1-dimensional finite topological complexes.

Spatial graphs $\mathcal{G}$ and $\mathcal{G}'$ are said to be equivalent if there is an ambient isotopy of S^3 which transforms $\mathcal{G}$ into $\mathcal{G}'$.

Even if an (abstract) graph is simple combinatorially, its embedding to $\mathbb{S}^3$ can be very complicated topologically.

Let Θ be a theta-graph and $\mathcal{G}$ be a spatial Θ-graph.



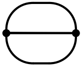
For e_1, e_2 and e_3, edges of a Θ, the images $K_1 = f(e_2 \cup e_3)$, $K_2 = f(e_1 \cup e_3)$, and $K_3 = f(e_1 \cup e_2)$ are said to be constitute knots of the spatial graph $\mathcal{G}$.

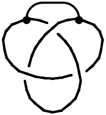
Theorem (K. Wolcott, 1986)

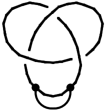
For any three given knots K_1, K_2, and K_3 there exists a spatial Θ-graph $\mathcal{G}$ such that these knots are realized as constitute knots of $\mathcal{G}$.
Moreover, knots K_1, K_2, and K_3 do not determine spatial Θ-graph uniquely.

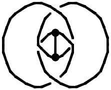
[Keith Wolcott, The knotting of Theta-curves and other graphs in S^3, Thesis, U. Iowa, 1986]

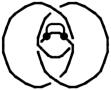
Θ-graphs with at most 5 crossings in a diagram

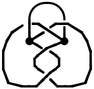

Θ_1

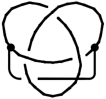

Θ_2

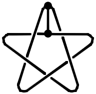

Θ_3

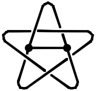

Θ_4

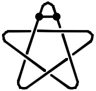

Θ_5

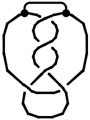

Θ_6

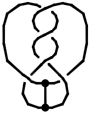

Θ_7

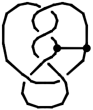

Θ_8

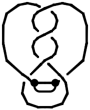

Θ_9


Θ_{10}


Θ_{11}


Θ_{12}


Θ_{13}

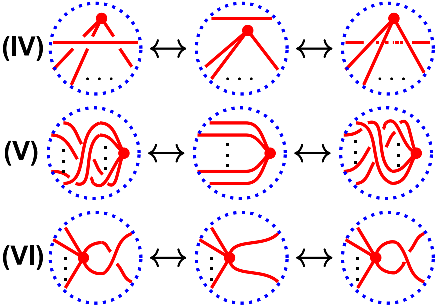

Θ_{14}

[J. Simon, A topological approach to the stereochemistry of nonrigid molecules, Graph theory and topology in chemistry, 1987.]

[H. Moriuchi, An enumeration of theta-curves with up to seven crossings, JKTR 2009, 18(2), 167-197.]

Theorem (L. Kauffman, S. Yamada, 1989)

Two spatial graphs $\mathcal{G}$ and $\mathcal{G}'$ in $\mathbb{S}^3$ are ambient isotopic if and only if a diagram of $\mathcal{G}$ can be transformed to a diagram of $\mathcal{G}'$ by a finite sequence of moves among (I)–(VI) and plane isotopy.



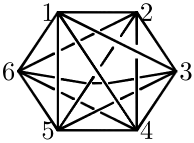
Existence of knots and links in a spatial complete graph

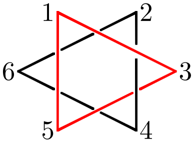
If a graph is "large enough" combinatorially, then any embedding is "knotted". Let $\mathbf{K_n}$ be a complete graph with n vertices. A graph is complete if any two vertices are connected by edge, so $\mathbf{K_n}$ has $\frac{n(n-1)}{2}$ edges.

Theorem (J. Conway, C. Gordon, 1983)

(1) Each embedding of $\mathbf{K_6}$ in $\mathbb{S}^3$ contains a pair of cycles which form an unsplittable 2-component link.
(2) Each embedding of $\mathbf{K_7}$ in $\mathbb{S}^3$ contains a cycle which is a non-trivial knot.

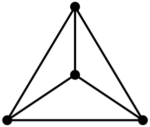
A spatial graph $\mathcal{G}$ is said to be splittable if there exists a 2-sphere S in $\mathbb{S}^3 \setminus \mathcal{G}$ which splits $\mathbb{S}^3$ into 3-balls B_1^3 and B_2^3 with both $B_1^3 \cap \mathcal{G}$ and $B_2^3 \cap \mathcal{G}$ nonempty. Otherwise $\mathcal{G}$ is said to be unsplittable.



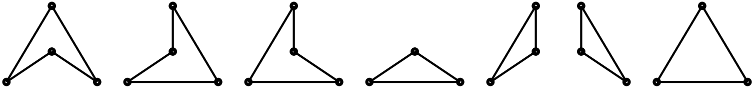


[J. Conway, C.McA. Gordon, Knots and links in spatial graphs, J. Graph Theory, 1983, 7, 445–453.]

Complete graph $\mathbf{K}_4$ and its embeddings



Graph $\mathbf{K_4}$.



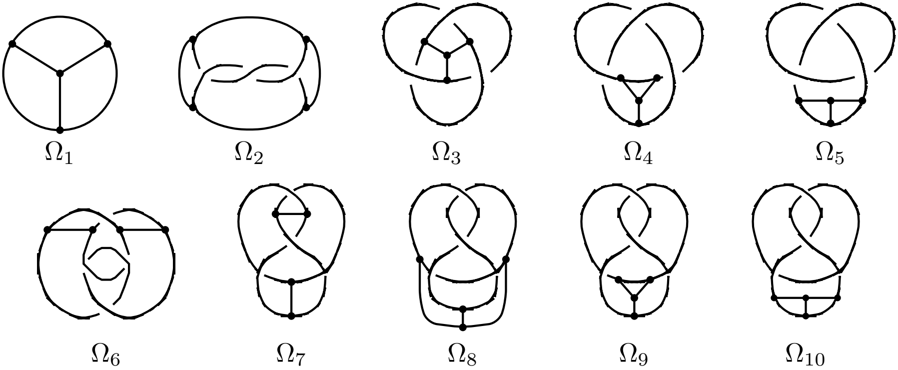
Seven cycles in $\mathbf{K_4}$.

Theorem (M. Yamamoto, 1990)

Let $c_1, \dots, c_7$ be the seven cycles in $\mathbf{K_4}$. For any ordered 7-tuple $(K_1, \dots, K_7)$ of knots, there is a spatial embedding of $\mathbf{K_4}$, such that cycles $(c_1, \dots, c_7)$ are embedded as knots $(K_1, \dots, K_7)$.

[M. Yamamoto, Knots in spatial embeddings of the complete graph on four vertices, Topology Appl., 1990, 36(3), 291–298.]

Table of spatial K_4-graphs with at most 4 crossings in a diagram

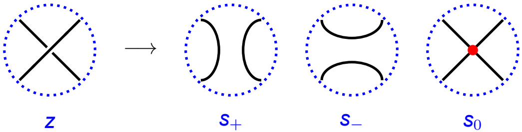


[J. Simon, A topological approach to the stereochemistry of nonrigid molecules, Graph theory and topology in chemistry, 1987.]

States of a spatial graph diagram

Let $\mathcal{G}$ be a spatial graph. Let D be a diagram of $\mathcal{G}$.

For any crossing $z \in D$ we define three local transformations of D: s_+-split, s_--split, and s_0-split:



After splitting in all crossings of D we get a state that is a plane graph.

Yamada polynomial $H(G)$ of a graph

Let $G = G(V, E)$ be a (combinatorial) graph, possibly with loops and multiply edges, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges.

Denote the number of connected components by $\omega(G)$ and the 1-st Betti number by

$$\beta(G) = |E(G)| - |V(G)| + \omega(G).$$

Definition

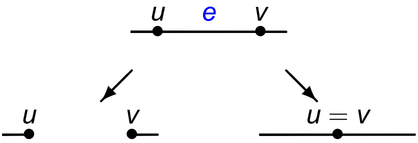
For a graph G define Laurent polynomial $H(G; A)$ in A by

$$H(G; A) = \sum_{F \subseteq E(G)} (-1)^{\omega(G-F)} (-A - 2 - A^{-1})^{\beta(G-F)},$$

where F passes over all subsets of $E(G)$.

Here $G - F$ is a graph with the vertex set $V(G)$ and the edge set $E(G) - F$.

Let $G = (V, E)$ be a finite graph. For edge $e \in E(G)$ denote by $G - e$ a graph obtained by deletion of e, and by G/e the graph, obtained by contraction of e.



The following properties of $H(G)$ hold.

1°. $H(G) = H(G/e) + H(G - e)$. (The same as for the Tutte polynomial!)
2°. If G and G' be homeomorphic graphs, then $H(G) = H(G')$.
3°. If $G \cup G'$ is a disjoint union of graphs, then $H(G \cup G') = H(G) \cdot H(G')$.
4°. If $G \cdot G'$ is a union along one vertex, then $H(G \cdot G') = -H(G) \cdot H(G')$.
5°. If G has a isthmus, then $H(G) = 0$.
6°. Let L_q be the one-vertex graph with q loops, then

$$H(L_q) = (-1)^{q+1}(A + 1 + A^{-1})^q.$$

Assume that diagram D of a spatial graph $\mathcal{G}$ has n crossings. Let S be a state of D, i.e., a plane graph, obtained by applying splits to all crossings of D.

Let $U(D)$ be the set of all states of D. Obviously, $|U(D)| = 3^n$.

Suppose that state S is obtained from D by applying $m_1(S)$ s_+-splits, $m_2(S)$ s_--splits, and so, by $(n - m_1(S) - m_2(S))$ s_0-splits.

Definition

Yamada polynomial $Y(D)$ of a spatial graph diagram D is a Laurent polynomial in A, defined by

$$Y(D) = Y(D; A) = \sum_{S \in U(D)} A^{m_1(S) - m_2(S)} H(S; A).$$

For the empty graph we suppose $Y(\emptyset) = 1$.
If diagram D of $\mathcal{G}$ has no crossings, then we get $Y(D) = H(\mathcal{G}; A)$.

[S. Yamada, An invariant of spatial graphs, J. Graph Theory, 1989, 13, 537–551.]

Maximal degree of a graph is the maximum of degres of its vertices, $\max\{deg(v) | v \in V(G)\}$.

Theorem (S. Yamada, 1989)

(1) If diagrams D and D' are equivalent under generalized Reidemeister moves (I) – (V), then $Y(D)$ and $Y(D')$ are equal up to a multiplier $(-A)^k$ for some integer k.

(2) Let D and D' be diagrams of spatial graphs of maximal degree at most 3. If diagrams D and D' are equivalent under generalized Reidemeister moves (I) – (VI), then $Y(D)$ and $Y(D')$ are equal up to a multiplier $(-A)^k$ for some integer k.

If D is a diagram of a spatial graph G we will denote $Y(D)$ by $Y(\mathcal{G})$.

[S. Yamada, An invariant of spatial graphs, J. Graph Theory, 1989, 13, 537–551.]

Yamada polynomials of spatial $\mathbf{K}_4$-graphs with at most 4 crossings

Yamada polynomials of above pictured spatial $\mathbf{K}_4$-graphs $\Omega_1, \ldots, \Omega_{10}$ are presented in the table.

Graph $\mathcal{G}$	Yamada polynomial $Y(\mathcal{G})$
Ω_1	$A^3+2A+2A^{-1}+A^{-3}$
Ω_2	$A^8+A^6+A^5-A^4+A^3-2A^2+A-1+A^{-1}+A^{-2}+A^{-3}+A^{-4}+A^{-5}$
Ω_3	$2A^6+A^4+A^3-2A^2-4-A^{-1}-3A^{-2}-A^{-3}+A^{-7}$
Ω_4	$A^8-A^7+A^6-A^4+A^3-2A^2+A-2-A^{-2}-A^{-3}-A^{-4}-A^{-6}$
Ω_5	$A^8-A^7+A^6-A^5-A^4-2A^2+A-1+2A^{-1}+A^{-2}+2A^{-3}+A^{-4}+2A^{-5}+A^{-7}$
Ω_6	$A^7-A^6+A^4+A^2+3A+3A^{-1}-A^{-2}+A^{-3}-A^{-4}-2A^{-5}+A^{-6}-A^{-7}+A^{-9}$
Ω_7	$-A^8-A^5+A^4+A^3+3A+3A^{-1}+A^{-3}+A^{-4}-A^{-5}-A^{-8}$
Ω_8	$A^9-A^8+2A^6-A^5+A^4+2A^3-A^2+2A-2+A^{-1}-A^{-2}-A^{-3}+2A^{-4}+2A^{-7}$
Ω_9	$-A^8+A^7-A^5+2A^4+2A-1+2A^{-1}-A^{-2}+A^{-3}+A^{-4}-A^{-5}+A^{-6}+A^{-7}-A^{-8}+A^{-9}$
Ω_{10}	$A^9-A^8+A^7-A^5+A^4+2A+2A^{-1}+A^{-4}-A^{-5}+A^{-7}-A^{-8}+A^{-9}$

[A. Vesnin, A. Dobrynin, The Yamada polynomial for graphs, embedded knot-wise into three-dimensional space, Vychisl. Sistemy, 155 (1996) 37–86. (in Russian). An English translation is available at https://www.researchgate.net/publication/266336562.]

Theorem (M. Li, F. Lei, F. Li, A. Vesnin, 2019)

The set of zeros of Yamada polynomial of all spatial graphs is dense in the complex plane $\mathbb{C}$.

The proof is constructive, the infinite family of spatial graphs with this property is constructed.

[M. Li, F. Lei, F. Li, A. Vesnin, On the Yamada polynomial of spatial graphs obtained by edge replacements, J. of Knot Theory and Ramifications, 2019, 27(9), 1842004.]

[M. Li, F. Lei, F. Li, A. Vesnin, Density of roots of the Yamada polynomial of spatial graphs, Proc. Steklov Institute of Mathematics, 2019, 305, 135–148.]

F. Jaeger [1997] introduced a Laurent polynomial invariant of a spatial graph $\mathcal{G}$ which can be calculated from its diagram.

We call this invariant Jaeger polynomial and denote it by $\mathfrak{J}(\mathcal{G}; A) \in \mathbb{Z}[A^{\pm 1}]$.

Y. Huh [2024] established the following relation between Jaeger polynomial and Yamada polynomial.

Lemma (Y. Huh, 2024)

Let G be a planar graph with vertices set $V(G)$ and edges set $E(G)$. Let D be a diagram of a spatial embedding of G. Then

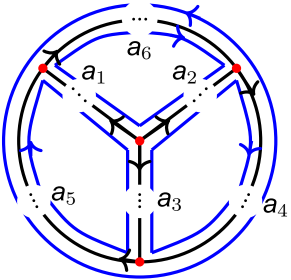
$$\mathfrak{J}(D; A) = \frac{Y(D; A^4)}{-(A^2 + A^{-2})^{|E(G)|-|V(G)|+1}}.$$

We will use this Lemma as a definition of $\mathfrak{J}(G; A)$ for $G = \mathbf{K_4}$.

[F. Jaeger, On some graph invariants related to the Kauffman polynomial, Progress in knot theory and related topics, 1997.]

[Y. Huh, Yamada polynomial and associated link of θ-curves, Discrete Mathematics, 2024, 347, paper number 113684]

Let $\mathcal{G}$ be a spatial embedding of $\mathbf{K_4}$, and D be a diagram of $\mathcal{G}$. Taking bands instead of edges we will get a band diagram which represents a three-punctured disk S, where D is a spine of S.



Denote $L = \partial S$, then L is a link in $\mathbb{S}^3$ with four components.

Seifert linking form

For an oriented surface S let x and y be closed curves on S. Let x^+ denote the result of pushing x a small amount into $\mathbb{S}^3 \setminus S$ along a positive normal to S. The function

$$\langle [x], [y] \rangle : H_1(S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

defined by $\langle [x], [y] \rangle = lk(x^+, y)$ is known as the Seifert form for S.

Theorem (KSWZ, 1993)

Let $\mathcal{G}_0$ be a planar embedding of a connected trivalent planar graph G. Suppose $\mathcal{G}_0$ is prime.

(1) If the number of edges in G is at most 6, then for each $\mathcal{G}$ there exists a unique (up to ambient isotopy) surface $S(\mathcal{G})$ with zero Seifert form.

(2) If the number of edges in G is more than 6, then

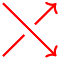
- (i) there exists a $\mathcal{G}$ with no $S(\mathcal{G})$ of zero Seifert form;
- (ii) if there is an $S(\mathcal{G})$ of zero Seifert form, it is the unique such surface.

In this sense, spatial $\mathbf{K_4}$-graphs are the largest «good» case.

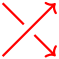
[L. Kauffman, J. Simon, K. Wolcott, P. Zhao, Invariants of theta-curves and other graphs in 3-space, Topology Appl., 1993]

Surface with zero Seifert form

For $i, j = 1, \dots, 6$ we denote by w_{ij} the sum of signs over all crossings of a_i and a_j, in particular, the sum over all self-crossings if $j = i$.



$\varepsilon(c) = +1$



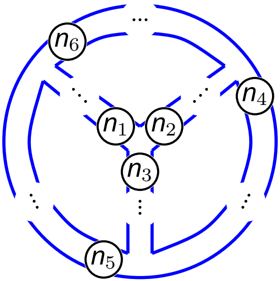
$\varepsilon(c) = -1$

[KSWZ]: To obtain a surface with zero Seifert form we start from a band diagram and apply additional twists with half-ntegers $n_1, \dots, n_6$ such that:

$$\begin{cases} n_1 = -w_{11} + \frac{1}{2}(-w_{23} - w_{25} + w_{21} + w_{13} + w_{15} + w_{36} - w_{16} + w_{56}); \\ n_2 = -w_{22} + \frac{1}{2}(-w_{24} + w_{14} + w_{46} + w_{23} - w_{13} - w_{36} + w_{12} + w_{26}); \\ n_3 = -w_{33} + \frac{1}{2}(w_{34} - w_{14} + w_{45} + w_{23} + w_{25} - w_{12} + w_{13} - w_{35}); \\ n_4 = -w_{44} + \frac{1}{2}(-w_{24} + w_{34} - w_{46} + w_{36} - w_{26} - w_{45} - w_{25} + w_{35}); \\ n_5 = -w_{55} + \frac{1}{2}(-w_{35} + w_{15} - w_{36} + w_{16} - w_{56} - w_{34} + w_{14} - w_{45}); \\ n_6 = -w_{66} + \frac{1}{2}(-w_{16} + w_{26} + w_{25} - w_{15} - w_{56} + w_{24} - w_{14} - w_{46}). \end{cases}$$

A link associated to diagram D

Taking above half-integers «twisting parameters» $n_1, \ldots, n_6$ we modify surface S by adding n_i-twists on bands.



Let $\mathcal{L} = L(n_1, \ldots, n_6)$ be a link obtained by applying n_i full twists as presented.

A link $\mathcal{L} = \partial S$ is said to be an associated link to diagram D.

Let D be a diagram of a spatial $\mathbf{K_4}$-graph $\mathcal{G}$.

Let $n_1, \dots, n_6$ be twisting parameters for D.

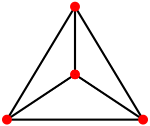
Define a normalized Jaeger polynomial;

$$\tilde{\mathfrak{J}}(D) = (-A)^{8(n_1+n_2+n_3+n_4+n_5+n_6)} \, \mathfrak{J}(D).$$

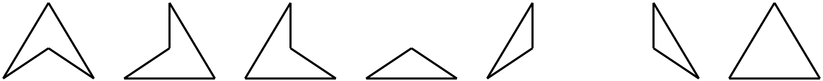
Theorem 1 [Oshmarina, V., 2024]

Let $\mathcal{G}$ be an embedding of $\mathbf{K_4}$ in $\mathbb{S}^3$, and D a diagram of $\mathcal{G}$. Then $\tilde{\mathfrak{J}}(D)$ is an invariant of $\mathcal{G}$.

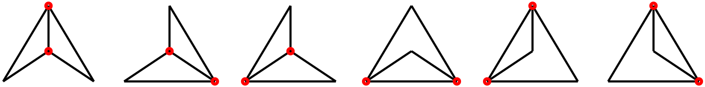
Knots and theta-subgraphs in $\mathbf{K_4}$



Consider embedding $\mathcal{G}$ of $\mathbf{K_4}$ in $\mathbb{S}^3$.



Then cycles of $\mathbf{K_4}$ are embedded as knots $K_1, \ldots, K_7$



and theta-subgraphs of $\mathbf{K_4}$ are embedded as spatial graphs $\Theta_1, \ldots, \Theta_6$.

For a link $L \subset \mathbb{S}^3$ denote its Jones polynomial by $V(L) \in \mathbb{Z}[A^{\pm 1}]$.

Theorem 2 [Oshmarina, V., 2024]

Let $\mathcal{G}$ be an embedding of $\mathbf{K_4}$ in $\mathbb{S}^3$. Denote

- by $\mathcal{L}$ the 4-component link associated to a diagram of $\mathcal{G}$,
- by $K_1, \dots, K_7$ knots which are embeddings of cycles of $\mathbf{K_4}$, and
- by $\Theta_1, \dots, \Theta_6$ spatial graphs which are embeddings of theta-subgraphs.

Then

$$\tilde{\mathfrak{J}}(\mathcal{G}) = V(\mathcal{L}) + \frac{1}{\varphi} \sum_{i=1}^{6} \tilde{\mathfrak{J}}(\Theta_i) - \frac{1}{\varphi^2} \sum_{j=1}^{7} \tilde{\mathfrak{J}}(K_j) + \frac{1}{\varphi^3},$$

where $\varphi = A^2 + A^{-2}$.

Corollary (Oshmarina, V., 2024)

Let $\mathcal{G}$ be an embedding of $\mathbf{K_4}$ in $\mathbb{S}^3$. Denote

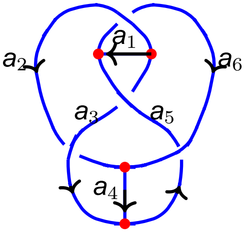
- by $\mathcal{L}$ the associated 4-component link to a diagram of $\mathcal{G}$,
- by $\mathcal{L}_i$, $i = 1, \dots, 6$ associated 3-component links to Θ-subgraphs of $\mathcal{G}$,
- by $K_j^{(2)}$, $j = 1, \dots, 7$, 2-component links which are 2-parallel to constitute knots of $\mathcal{G}$ with some additional full-twists.

Then

$$\widetilde{\mathfrak{J}}(\mathcal{G}) = V(\mathcal{L}) + \frac{1}{\varphi} \sum_{i=1}^{6} V(\mathcal{L}_i) + \frac{2}{\varphi^2} \sum_{j=1}^{7} V(K_j^{(2)}) + \frac{6}{\varphi^3}.$$

where $\varphi = A^2 + A^{-2}$.

Spatial $\mathbf{K_4}$-graph Ω_7.



Spatial $\mathbf{K_4}$-graph Ω_7.

The normalized Jeager polynomial:

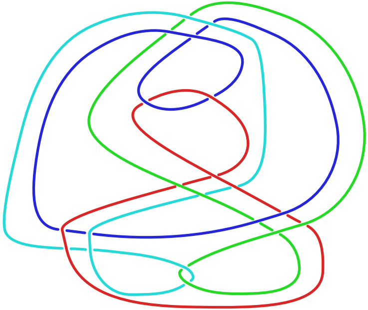
$$\widetilde{\mathfrak{J}}(\Omega_7) = \frac{-1}{\varphi^3}(-A^{32} - A^{20} + A^{16} + A^{12} + 3A^4 + 3A^{-4} + A^{-12} - A^{-20} - A^{-32}),$$

where $\varphi = A^2 + A^{-2}$.

Twisting parameters are $n_1 = -1$, $n_4 = 1$, and $n_i = 0$ for $i = 2, 3, 5, 6$.

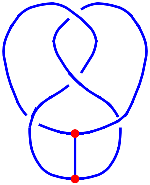
An example for Theorem 2 (part II)

The associated link $\mathcal{L}$ to the spatial $\mathbf{K_4}$-graph Ω_7:

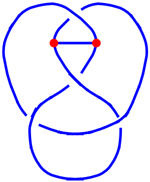


- Jones polynomial $V(\mathcal{L}) = A^{30} - 2A^{26} + A^{22} + A^{18} - 3A^{14} + 3A^{10} - 3A^{6} - 2A^{2} - 2A^{-2} - 3A^{-6} + 3A^{-10} - 3A^{-14} + A^{-18} + A^{-22} - 2A^{-26} + A^{-30}$.
- The complement $\mathbb{S}^3 \setminus \mathcal{L}$ admits a complete Riemann metric of curvature -1. The number $\mathrm{vol}(\mathbb{S}^3 \setminus \mathcal{L}) = 18.9807764741\ldots$ is an invariant of Ω_7 under an ambient isotopy.

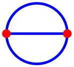
Six theta-subgraphs $\Theta_1, \ldots, \Theta_6$.



Theta-subgraph Θ_1.



Theta-subgraph Θ_4.



Theta-subgraphs $\Theta_2, \Theta_3, \Theta_5, \Theta_6$.

$$\widetilde{\mathfrak{J}}(\Theta_1) = \frac{1}{\varphi^2}(-A^{36} + A^{28} + A^{20} + A^{16} + A^{8} + 1 + A^{-12} + A^{-24})$$

$$\widetilde{\mathfrak{J}}(\Theta_4) = \frac{1}{\varphi^2}(A^{24} + A^{12} + 1 + A^{-8} + A^{-16} + A^{-20} + A^{-28} - A^{-36})$$

$$\widetilde{\mathfrak{J}}(\Theta_i) = \frac{1}{\varphi^2}(A^{8} + A^{4} + 2 + A^{-4} + A^{-8}) \quad \text{for} \quad i \in \{2, 3, 5, 6\}.$$

Seven constituent knot $K_1, \dots, K_7$.

The knot K_1 formed by set of edges $\{2, 3, 5, 6\}$ is the figure-eight knot 4_1, other knots $K_2, \dots, K_7$ are trivial knots. Therefore,

$$\widetilde{\mathfrak{J}}(K_1) = -A^{26} + A^{22} - A^{2} - A^{-2} + A^{-22} - A^{-26} + \frac{1}{\varphi}.$$

and

$$\widetilde{\mathfrak{J}}(K_2) = \widetilde{\mathfrak{J}}(K_3) = \widetilde{\mathfrak{J}}(K_4) = \widetilde{\mathfrak{J}}(K_5) = \widetilde{\mathfrak{J}}(K_6) = \widetilde{\mathfrak{J}}(K_7) = -A^{2} - A^{-2} + \frac{1}{\varphi}.$$

Thus, the following relation holds for $\Omega = \Omega_7$:

$$\widetilde{\mathfrak{J}}(\Omega) - V(\mathcal{L}) = \frac{1}{\varphi} \sum_{i=1}^{6} \widetilde{\mathfrak{J}}(\Theta_i) - \frac{1}{\varphi^2} \sum_{j=1}^{7} \widetilde{\mathfrak{J}}(K_j) + \frac{1}{\varphi^3}.$$

Thank you for your attention!