Flag-transitive 3-design from the action of $\mathrm{PSL}(2,q)$ on the projective line

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Let q be a prime power. It is well-known that $\mathrm{PSL}(2,q)$ acts on the projective line $\mathrm{PG}(1,q)=\mathbb{F}_q\cup\{\infty\}$ via linear fractional transformations:

$$f(z)=\frac{az+b}{cz+d},\quad \text{where } z\in\mathbb{F}_q\cup\{\infty\},\ \begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{PSL}(2,q).$$

If $q\equiv 1 \pmod 4$, then there are exactly two $\mathrm{PSL}(2,q)$-orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ on 3-element subsets of $\mathbb{F}_q\cup\{\infty\}$, with representatives $T_+=\{\infty,0,1\}$ and $T_-=\{\infty,0,\alpha\}$, respectively, where α is a primitive element of $\mathbb{F}_q$.

Bonnecaze and Solé [1] found that the extended quadratic residue code of length 42 supports a (seemingly sporadic) 3-$(42,10,18)$ design. It turns out that this design has $\mathrm{PSL}(2,41)$ as a flag-transitive automorphism group, and has the multiplicative subgroup of 10th roots of unity in $\mathbb{F}_{41}$ as a starter block.

The purpose of this talk is to show that this 3-$(42,10,18)$ design is the first one in the family of flag-transitive 3-$(q+1,10,18)$ designs, where q is an odd power of a prime in the sequence A325072 in OEIS:

$$41,61,241,281,421,601,641,\dots.$$

This sequence consists of primes p satisfying $p\equiv 1 \pmod{20}$ and one of the following equivalent conditions:

(i) there exists $\theta\in\mathbb{F}_p^\times\setminus(\mathbb{F}_p^\times)^2$ such that $\theta^2-4\theta-1=0$,

(ii) $p\neq x^2+20y^2$ for any integers x,y,

(iii) $p\neq x^2+100y^2$ for any integers x,y,

(iv) 5 is a not a quartic residue in $\mathbb{F}_p$.

Li, Deng and Zhang [2], show that if p satisfies (i) above, then the orbit of $\{1,\beta,\beta^2,\beta^3,\beta^4\}$ under $\mathrm{PSL}(2,p)$ is a flag-transitive 3-$(p+1,5,3)$ design. Moreover, they showed that p can be a prime power, not necessarily a prime, as long as condition (i) is satisfied.

Here is our main result.

Theorem 1 *Suppose that p is a prime with $p\equiv 1 \pmod{20}$ satisfying one of the equivalent conditions* (i)–(iv) *above, and let α be a primitive 10th root of unity in $\mathbb{F}_p$. If q is an odd power of p, then the orbit of $\{1,\beta,\beta^2,\dots,\beta^9\}$ under $\mathrm{PSL}(2,q)$ is a flag-transitive 3-$(q+1,10,18)$ design.*

References

[1] A. Bonnecaze and P. Solé. The extended binary quadratic residue code of length 42 holds a 3-design. *J. Combin. Des.*, **29** (2021) 528–532.

[2] Weixia Li, Dameng Deng, and Guangjun Zhang, Simple 3-$(q+1,5,3)$ designs admitting an automorphism group $\mathrm{PSL}(2,q)$ with $q\equiv 1 \pmod 4$. *Ars Combin.*, **136** (2018) 97–108.