Exact results on traces of sets

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This is a joint work with Mingze Li and Mingyuan Rong.

For non-negative integers n, m, a and b, we write $(n, m) \to (a, b)$ if for every family $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geqslant m$ there is an a-element set $T \subseteq [n]$ such that $\left|\mathcal{F}_{|T}\right| \geqslant b$, where $\mathcal{F}_{|T} = \{F \cap T : F \in \mathcal{F}\}$. A longstanding problem in extremal set theory asks to determine $m(s) = \lim_{n\to+\infty} \frac{m(n,s)}{n}$, where $m(n, s)$ denotes the maximum integer m such that $(n, m) \to (n - 1, m - s)$ holds for non-negatives n and s. In this paper, we establish the exact value of $m(2^{d-1} - c)$ for all $1 \leqslant c \leqslant d$ whenever $d \geqslant 50$, thereby solving an open problem posed by Piga and Schülke. To be precise, we show that

$$m(n, 2^{d-1} - c) = \begin{cases} \frac{2^d - c}{d} n & \text{for } 1 \leq c \leq d - 1 \text{ and } d \mid n \\ \frac{2^d - d - 0.5}{d} n & \text{for } c = d \text{ and } 2d \mid n \end{cases}$$

holds for $d \geq 50$. Furthermore, we provide a proof that confirms a conjecture of Frankl and Watanabe from 1994, demonstrating that $m(11) = 5.3$.