Invariants of graphs, embedded graphs, delta-matroids, and permutations

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Lectures 5 and 6: Knot invariants and weight systems

1. Knots and their invariants
2. Plane diagram of a knot
3. Reidemeister moves
4. Vassiliev knot invariants
5. Chord diagrams and 4-term relations for them
6. Weight systems

Lectures 5 and 6: Knot invariants and weight systems: knots

A *knot* is a nondegenerate smooth embedding $S^1 \to S^3$ of the circle to the 3-sphere. Two knots are said to be *ambient isotopic* if there are two smooth families of orientation-preserving diffeomeorphisms $S^1 \to S^1$ of the source and $S^3 \to S^3$ of the target, respectively, depending on a single parameter t such that they deform the first knot at $t = 0$ to the second one at $t = 1$.

This definition has a variety of versions, but for our current purposes it is sufficient.

Lectures 5 and 6: Knot invariants and weight systems: Knot presentation

There is a number of ways to represent knots, most of them suitable for computer presentation. The most common way to represent a knot is by means of a *plane diagram*, which is a generic projection of a knot to a two-dimensional plane having finitely many points of transversal double intersection, with overpassing/underpassing of the strands at the double point indicated.

Lectures 5 and 6: Knot invariants and weight systems: Knot presentation

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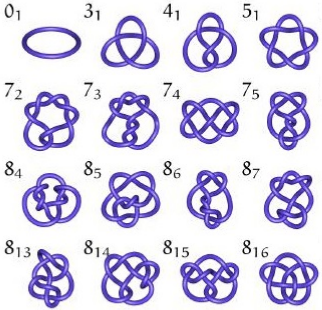
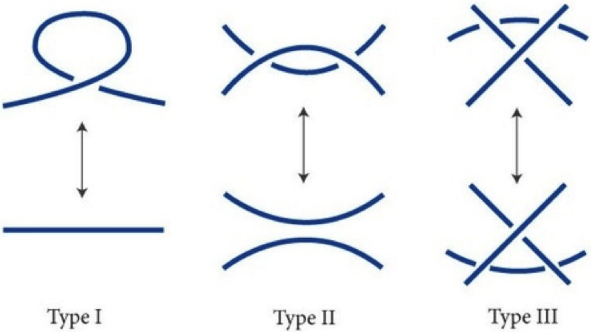


Lectures 5 and 6: Knot invariants and weight systems: Reidemeister moves

Different knot diagrams can represent one and the same knot. In particular, it is clear that any two diagrams related by a sequence of *Reidemeister moves* represent one and the same knot.



Theorem (K. Reidemeister (1927))

If two knot diagrams represent the same knot, then they are related by a sequence of Reidemeister moves.

Lectures 5 and 6: Knot invariants and weight systems: Reidemeister moves

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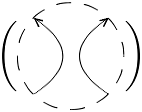
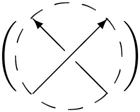
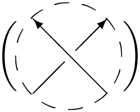
As a consequence, one can define a knot invariant in terms of a plane knot diagram, and prove that it remains unchanged under each Reidemeister move.

Lectures 5 and 6: Knot invariants and weight systems: Conway polynomial

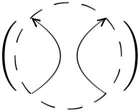
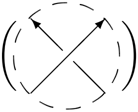
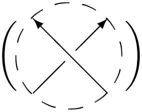
Conway polynomial is a knot invariant with values in the ring $\mathbb{Z}[x]$ of polynomials in one variable x. In fact, the Conway polynomial is defined on *links*, not only on knots. A *link* is an isotopy class of nonsingular embeddings of a finite set of circles in $\mathcal{R}^3$.
Consider a generic plane projection of a knot (or a link), i.e., a projection with only transversal double points of self-intersection. We define the Conway polynomial by the recurrence rule:

$x \cdot \mathrm{Con}$  $=$ Con  $-$ Con 

and the initial value $\mathrm{Con}(\mathrm{unknot}) = 1$.
This definition allows one to calculate the value of the Conway polynomial knowing a generic projection of the knot.

Lectures 5 and 6: Knot invariants and weight systems: HOMFLY polynomial

The *HOMFLY polynomial* $H = H(N, q)$ is a knot (and link) invariant. It depends on two variables, denoted usually by N and $q^{1/2}$. Similarly to the Conway polynomial, the HOMFLY polynomial is defined by the relation

$(q^{1/2} - q^{-1/2})H$  $= q^{N/2}H$  $- q^{-N/2}H$ 

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Lectures 5 and 6: Knot invariants and weight systems: Vassiliev knot invariants

Classical knot invariants are usually defined in terms of plane knot diagrams. One defines a function on knot diagrams and then proves it is invariant with respect to Reidemeister moves of three types.

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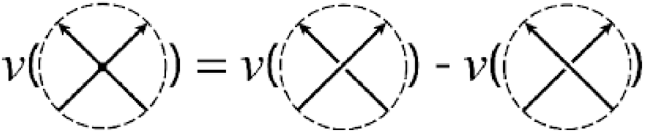
Definition

A *singular knot* is an *immersion* $S^1 \to S^3$, that is, a smooth nondegenerate mapping such that each point in its image has at most two preimages, and if there are two, then the corresponding tangent vectors are not collinear.

Lectures 5 and 6: Knot invariants and weight systems: Extending invariant to singular knots

Let v be a knot invariant taking values in a commutative ring. By an *extension of v* to singular knots we mean an invariant v of singular knots that coincides with v on knots and satisfies *Vassiliev skein relation*:



Lectures 5 and 6: Knot invariants and weight systems: Extending invariant to singular knots

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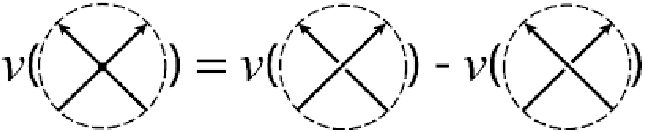

Theorem

Any knot invariant v admits a unique extension to singular knots.

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Proof. The proof proceeds by induction on the number n of double points in a singular knot. If $n = 0$, the knot is nonsingular, which provides induction base. Suppose we have proved the theorem for all singular knots with up to n double points. For a singular knot with $n + 1$ double points, take any of them. Then the value of v on this singular knot is uniquely determined by means of Vassiliev's skein relation applied to this double point: both singular knots on the right have n double points, whence the value of v on them is well-defined by the induction hypothesis. Finally, the result does not depend on the double point we have chosen: for another double point, the simultaneous resolution of both of them provides the same linear combination of values of v on four singular knots with $n - 1$ double points.

Lectures 5 and 6: Knot invariants and weight systems: finite type invariants

The following introduces the key notion of Vassiliev's theory.

Definition

A knot invariant is said to be *of order at most n* if its extension vanishes on all singular knots having at least $n+1$ double points. A knot invariant *is of finite type* if it is of order at most n, for some nonnegative integer n.

Finite type invariants with values in a given commutative ring form a commutative ring with respect to pointwise operations. V. Vassiliev conjectured that any two nonisotopic knots can be distinguished by a finite type knot invariant. This conjecture, however, is far from being proved or disproved.

Lectures 5 and 6: Knot invariants and weight systems: Conway polynomial

Theorem

Let C_n be the knot invariant equal to the coefficient of x^n in the Conway polynomial. Then C_n is a Vassiliev knot invariant of order $\leq n$.

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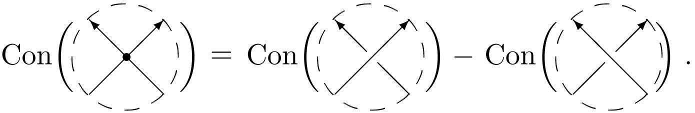
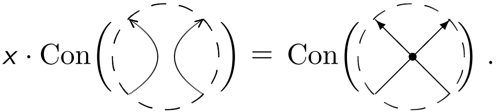
Indeed, extend the Conway polynomial to singular knots according to the rule



Then, by the defining recurrence rule we have



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Lectures 5 and 6: Knot invariants and weight systems: Conway polynomial

Therefore, the Conway polynomial of a singular knot with k double points is divisible by at least x^k, and the coefficient C_n obviously vanishes on singular knots with more than n double points.
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Although the coefficients of the Conway polynomial are Vassiliev invariants, the polynomial itself is not a Vassiliev invariant. Indeed, the sequence of its coefficients C_n has increasing order.

Lectures 5 and 6: Knot invariants and weight systems: chord diagrams

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Theorem

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Lectures 5 and 6: Knot invariants and weight systems: chord diagrams

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Theorem

A knot invariant of order at most n takes the same value on any two singular knots with n singular points having isomorphic chord diagrams.

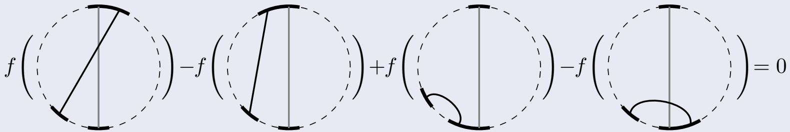
It is easy to prove that any chord diagram can appear as the chord diagram of a singular knot. In Vassiliev's theory, chord diagrams are considered as indicators of strata in a stratification of the space of singular knots: to each chord diagram, a stratum is associated, which consists of singular knots with this chord diagram.

Lectures 5 and 6: Knot invariants and weight systems: Four-term relations

The theorem above means that a knot invariant of order at most n determines a function on chord diagrams with n chords. A natural question arises: whether any function on chord diagrams can be obtained in this way? V. Vassiliev gave a negative answer to this question:

Theorem

Any function on chord diagrams of order n arising from a knot invariant of order at most n satisfies the four-term relations:

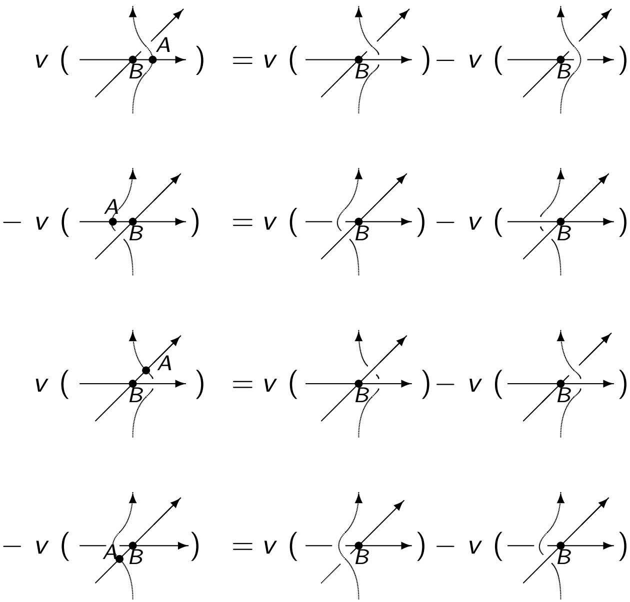


Here all four diagrams, in addition to the two chords in the pictures, can have an arbitrary set of chords, which is one and the same in all of them.

Lectures 5 and 6: Knot invariants and weight systems: Four-term relations

In order to prove the four-term relations, consider the 'fictional character', which is the triple point singularity of a knot, and four singular knots close to such a knot with a triple point.

Each of these four singular knots have a 'stationary' double point B, which is the intersection of two horizontal strands, and a 'moving' double point A, which is the intersection point of the vertical strand with one of the four horizontal strands. The four chord diagrams of these four singular knots are exactly those entering the four-term relation. The relation is proved by applying Vassiliev's skein relation to each of the four moving double points A and noting that each term enters the alternating sum of the four terms exactly twice, one times with $+$ and the other time with $-$ sign. This proves the relation.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M. Kontsevich (1993) proved the inverse statement, which is much harder:

Theorem

Each function on chord diagrams satisfying four-term relations arises from some finite type knot invariant.

Functions on chord diagrams satisfying four-term relations are called *weight systems*.

A chord diagram is nothing but a map with a single vertex (represented by the outer circle). Therefore, it is worth to analyze, which invariants of maps satisfy four-term relations.

Lectures 5 and 6: Knot invariants and weight systems: First examples

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Example

The number of faces of a chord diagram satisfies four-term relations, whence is a weight system.

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Example

The chromatic polynomial of the intersection graph of a chord diagram satisfies four-term relations, whence is a weight system.

Lectures 5 and 6: Knot invariants and weight systems: Problems

- Prove that a knot invariant of order at most n takes the same value on any two singular knots with isomorphic chord diagrams.
- Make the substitution $q = q(x) = 1 + q_1 x + q_2 x^2 + \dots$ to the HOMFLY polynomial. Prove that the coefficient of x^n in the power series $H(N, q(x))$ is a knot invariant of order at most n.
- Prove that the value of C_n, the coefficient of x^n of the Conway polynomial, on a chord diagram of order n is 1 if this chord diagram has a single face, and it is 0 otherwise.

**Thank you
for your attention**