Invariants of graphs, embedded graphs, delta-matroids, and permutations

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Lectures 3 and 4: Delta-matroids

1. Definition of a delta-matroid
2. Examples
3. Associating a delta-matroid to a graph
4. Associating a delta-matroid to an embedded graph
5. Duality and partial duality
6. Invariants of delta-matroids

Lectures 3 and 4: Delta-matroids: Definition

A *set system* is a finite set $\mathcal{E}$ together with a nonempty set $\Phi \subset 2^{\mathcal{E}}$ of subsets in it. Two set systems $D_1 = (\mathcal{E}_1; \Phi_1)$ and $D_2 = (\mathcal{E}_2; \Phi_2)$ are *isomorphic* if there is a one-to-one map $\mathcal{E}_1 \to \mathcal{E}_2$ taking Φ_1 one-to-one to Φ_2. Below, we consider set systems up to isomorphism.

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A *delta-matroid* is a set system $(\mathcal{E}; \Phi)$ possessing the following property: *for any pair $\phi, \psi \in \Phi$, and any element $e \in \phi \Delta \psi$ there is an element $e' \in \phi \Delta \psi$ such that $\phi \Delta \{e, e'\} \in \Phi$.*
Here Δ denotes the symmetric difference of two sets:
$A \Delta B = (A \setminus B) \sqcup (B \setminus A)$.

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Here Δ denotes the symmetric difference of two sets:
$A \Delta B = (A \setminus B) \sqcup (B \setminus A)$.
The set $\mathcal{E}$ is called the *ground set* of the delta-matroid. The elements of the set Φ are called the *feasible sets* of the delta-matroid.

Lectures 3 and 4: Delta-matroids: Definition

Example

The set system $(\{1\}; \{\emptyset, \{1\}\})$ is a delta-matroid. Indeed, for $\phi = \emptyset, \psi = \{1\}$ and $e = 1 \in \phi \Delta \psi = \{1\}$ we may take $e' = e = 1$, so that $\phi \Delta \{e, e'\} = \phi \Delta \{1\} = \psi$, and similarly if we exchange ϕ and ψ.

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Example

The set system $(\{1,2,3\};\{\emptyset,\{1,2,3\}\})$ is not a delta-matroid. Indeed, for $\phi=\emptyset,\psi=\{1,2,3\}$, and $e=1\in\phi\Delta\psi=\{1,2,3\}$, there is no $e'\in\phi\Delta\psi=\{1,2,3\}$ such that $\phi\Delta\{e,e'\}\in\Phi$.

There is a natural way to associate a delta-matroid to a graph.
A graph G is said to be *nondegenerate* if its adjacency matrix $A(G)$ is nondegenerate over the field of two elements (that is, if $\det A(G) = 1$).

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For a graph G, the associated delta-matroid D_G is defined as follows:

- the ground set $\mathcal{E}_G$ of D_G is $V(G)$, the set of vertices of G;
- the set Φ_G of admissible subsets of $V(G)$ consists of subsets ϕ such that the graph $G|_\phi$ is nondegenerate;
- by convention, the empty graph is nondegenerate.

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Theorem (A. Bouchet (1980))

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Example

For the tree A_3 on 3 vertices $1, 2, 3$ (which has edges $\{1, 2\}$ and $\{2, 3\}$), the corresponding delta-matroid D_{A_3} is

$$D_{A_3} = (\{1, 2, 3\}, \{\emptyset, \{1, 2\}, \{2, 3\}\}).$$

Lectures 3 and 4: Delta-matroids: delta-matroids of graphs on surfaces

There is a natural way to associate a delta-matroid to an embedded graph. An embedded graph Γ is called a *quasitree* if it has a single face. (The notion comes from the genus 0 case: a graph embedded into the sphere has a single face if and only if it is a tree.)

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For an embedded graph Γ, the associated delta-matroid D_Γ is defined as follows:

- the ground set $\mathcal{E}_\Gamma$ of D_Γ is $E(\Gamma)$, the set of edges of Γ;
- the set Φ_Γ of admissible subsets of $E(\Gamma)$ consists of subsets ϕ such that the embedded graph $\Gamma|_\phi$ is a quasitree;
- the embedded graph without edges is a quasitree if and only if it has a single vertex.

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Theorem (A. Bouchet (1980))

The set system D_Γ for an embedded graph Γ is indeed a delta-matroid.

Example

For the embedded graph Γ obtained by gluing the torus from the square, on two edges $1, 2$, the corresponding delta-matroid D_Γ is

$$D_\Gamma = (\{1, 2\}; \{\emptyset, \{1, 2\}\}).$$

A delta-matroid is *even* if the numbers of elements in all its feasible sets have the same parity (either are all even or all odd).

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For any graph G, its delta-matroid D_G is even. Indeed, a graph G can be nondegenerate only if it contains even number of vertices.
For an embedded graph Γ, its delta-matroid D_Γ is even if and only if Γ is orientable.

The two constructions of delta-matroids, the one for graphs and the one for embedded graphs, seem inharmonious. Indeed, for a graph G, the ground set of its delta-matroid D_G is the set of *vertices* $V(G)$ of G. In contrast, for an embedded graph Γ the ground set of the delta-matroid D_Γ is the set $E(\Gamma)$ of *edges* of Γ. The following argument shows, however, that the two definitions are consistent.

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Let us restrict ourselves with orientable embedded graphs having a single vertex. An edge in such an embedded graph is a loop, which connects the vertex with itself. We say that two loops *intersect* one another if their ends follow the boundary of the vertex in the alternating order. The *intersection graph* $\gamma(\Gamma)$ of an embedded graph Γ with a single vertex is the graph whose set of vertices is in one-to-one correspondence with the set of edges of Γ, and two vertices of $\gamma(\Gamma)$ are connected by an edge if and only if the corresponding edges intersect.

Theorem

For an orientable embedded graph Γ with a single vertex, the two delta-matroids D_Γ and $D_{\gamma(\Gamma)}$ are naturally isomorphic to one another.

In other words, an orientable embedded graph Γ with a single vertex has a single face iff its intersection graph $\gamma(\Gamma)$ is nondegenerate.

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The delta-matroid of an arbitrary embedded graph plays the role of intersection graph of the embedded graph with a single vertex, that is why *edges* of an embedded graph are a natural substitute for *vertices* of an abstract graph.

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Theorem

The delta-matroid $D_{\overline{\Gamma}}$ has the form

$$D_{\overline{\Gamma}} = \mathcal{E}_\Gamma * D_\Gamma = (\mathcal{E}_\Gamma; \{\phi \Delta \mathcal{E}_\Gamma | \phi \in \Phi_\Gamma\}).$$

In other words, feasible sets in $D_{\overline{\Gamma}}$ are complements to feasible sets in Γ: $\phi \Delta \mathcal{E}_\Gamma = \mathcal{E}_\Gamma \setminus \phi$.

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In other words, feasible sets in $D_{\overline{\Gamma}}$ are complements to feasible sets in Γ: $\phi \Delta \mathcal{E}_\Gamma = \mathcal{E}_\Gamma \setminus \phi$.
Recall that the set of edges $E(\overline{\Gamma})$ of the dual embedded graph is naturally isomorphic to that of Γ, which allows one to identify the ground sets of the delta-matroids D_Γ and $D_{\overline{\Gamma}}$.

The duality with respect to the ground set $\mathcal{E}$ of a delta-matroid can be extended to duality with respect to its arbitrary subset.
For a subset $F \subset \mathcal{E}$ of the ground set $\mathcal{E}$ of a delta-matroid $D = (\mathcal{E}; \Phi)$, we set $F * D = (\mathcal{E}; F * \Phi) = (\mathcal{E}; \{\phi \Delta F | \phi \in \Phi\})$ and call $F * D$ the *twist*, or the *partial dual of* D*, with respect to* F.

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Theorem

For an arbitrary subset $F \subset \mathcal{E}$ of the ground set $\mathcal{E}$ of a delta-matroid $D = (\mathcal{E};\Phi)$, the twist $F * D$ is a delta-matroid.

Clearly, it suffices to prove the theorem for F consisting of each single element $e \in \mathcal{E}$. Indeed, for an arbitrary $F = \{e_1, e_2, \dots\}$, we have $F * D = \dots * \{e_2\} * \{e_1\} * D$, so that twisting with respect to an arbitrary set can be considered as a composition of twistings with respect to one-element sets. The latter twistings commute with one another.
Partial duals of delta-matroids of abstract graphs are *binary*.

Lectures 3 and 4: Delta-matroids: Partial duality for embedded graphs

For an embedded graph Γ, the twist of its delta-matroid D_Γ with respect to its ground set $E(\Gamma)$ corresponds to replacing Γ with its dual $\overline{\Gamma}$. Whether there is a natural transformation of Γ corresponding to the twist with respect to an arbitrary given subset $F \subset E(\Gamma)$? Once again, it suffices to construct such a transformation for an arbitrary $F = \{e\}, e \in E(\Gamma)$.

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Definition

For an edge $e \in E(\Gamma)$ of an embedded graph Γ we set $\{e\} * \Gamma$ to be

- the result of replacing two ends of e with a single vertex, e with a loop connecting this vertex to itself, separating the chord ends belonging to the former vertices, if e connects two different vertices;
- the result of splitting the end vertex of e into two vertices, e connecting two new vertices, its ends attached to the same boundary segments, if e is an orientable ribbon connecting a vertex to itself.

In all the cases $\{e\} * \{e\} * \Gamma = \Gamma$, so that $\{e\}*$ is an involution.

For an arbitrary set system $D = (\mathcal{E}; \Phi)$, define the *distance to D* function d_D on the set $2^{\mathcal{E}}$ of subsets of the ground set by

$$d_D(F) = \min_{\phi \in \Phi} |F \Delta \phi|.$$

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Theorem

For the delta-matroid D_Γ determined by an embedded graph Γ, and a subset $F \subset E(\Gamma)$ of its edges, the distance from F to D_Γ is 1 less than the number of faces of the embedded graph $\Gamma|_F$.

Isomorphic graphs, as well as isomorphic embedded graphs, have isomorphic delta-matroids. Therefore, invariants of delta-matroids provide simultaneously invariants of graphs and embedded graphs. Many of graph and embedded graph invariants can be defined more naturally in the language of delta-matroids.

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Example

Define the *number of faces* $f(D)$ of a delta-matroid $D = (\mathcal{E}; \Phi)$ as the distance from the ground set $\mathcal{E}$ to D increased by 1:

$$f(D) = d_D(\mathcal{E}) + 1 = \min_{\phi \in \Phi} |\phi \Delta \mathcal{E}| + 1.$$

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For the delta-matroid of an embedded graph, the number of faces coincides with that of the embedded graph itself.
For the delta-matroid of a graph G, the number of faces coincides with the corank of the adjacency matrix $A(G)$ over the field of two elements increased by 1.

The *interlace polynomial* $Q_D(x)$ of a delta-matroid D is defined by

$$Q_D(x) = \sum_{F \subset \mathcal{E}} x^{d_D(F)}.$$

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An element e of delta-matroid is a *loop* if it does not enter any feasible set; e is a *coloop* if it enters each feasible set.

Theorem

Let $D = (\mathcal{E}, \Phi)$ be a delta-matroid with an element $e \in \mathcal{E}$ that is neither a coloop nor a loop. Then we have

$$Q_D(x) = Q_{D\setminus e}(x) + Q_{(D*e)\setminus e}(x).$$

Originally, the interlace polynomial of graphs was defined recursively. In order to introduce it, we firstly define the pivot operation for graphs. Let G be a graph, and let a and b be its two vertices connected by an edge. All the vertices except for a and b are divided into four classes:

1. vertices adjacent to both a and b;
2. vertices adjacent to a but not to b;
3. vertices adjacent to b but not to a;
4. vertices adjacent neither to a nor to b.

The *pivot* G^{ab} is the graph obtained from G by removing all the edges connecting vertices from two different classes among 1–3 above and adding such edges if they are not present in G. For chord diagrams and delta-matroids, pivot acts as partial duality with respect to the subset $\{a, b\}$ of the base set. On the delta-matroid $D_G(V(G); \Phi(G))$ of a graph G, the pivot acts as the partial duality with respect to the subset $\{a, b\} \subset V(G)$. In particular, the interlace polynomial of graphs is invariant under pivot.

The interlace polynomial $L_G(x)$ is defined by the initial condition
if there are no edges in the graph G, then $L_G(x) = (x+1)^n$ where n is the number of vertices in G
and the following recurrence:
if vertices a and b of the graph G are connected by an edge, then

$$L_G(x) = L_{G\setminus a}(x) + L_{G^{ab}\setminus b}(x), \tag{1}$$

where $G \setminus a$ is the graph obtained from G by erasing the vertex a and all the edges connecting a with other vertices.
In some papers the interlace polynomial is normalized differently; namely, it is defined as $L_G(x-1)$ in our notation; in other words, the initial condition for the discrete graph on n vertices is x^n rather than $(x+1)^n$.
The interlace polynomial admits a two-variable extension, which is defined for delta-matroids as

$$\overline{Q}_D(x,y) = \sum_{F\subset\mathcal{E}} y^{|F|}x^{d_D(F)}.$$

Lectures 3 and 4: Delta-matroids: Problems

- For which values of n the complete graph K_n is nondegenerate?
- For which values of n, m the complete bipartite graph $K_{m,n}$ is nondegenerate?
- Compute the interlace polynomial of the complete graph K_n.
- Compute the interlace polynomial of your favorite embedded graph.

- Prove that delta-matroids of embedded graphs are binary.
- Prove that the interlace polynomial of a delta-matroid is invariant under partial duality, so that partially dual delta-matroids have the same interlace polynomial.
- Let $\gamma(\Gamma)$ be the intersection graph of an embedded graph with a single vertex. Prove that the number of faces $f(\Gamma)$ of Γ is $\mathrm{corank}\ A(\gamma(\Gamma))+1$.

Thank you for your attention