Invariants of graphs, embedded graphs, delta-matroids, and permutations

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Lecture 8: Constructing weight systems from Lie algebras

1. Constructions
2. $\mathfrak{sl}(2)$- and $\mathfrak{gl}(N)$-weight systems
3. $\mathfrak{gl}(N)$-weight system for permutations
4. The universal $\mathfrak{gl}$-weight system
5. Inducing graph invariants from the universal $\mathfrak{gl}$-weight system

Initial data: finite dimensional Lie algebra $\mathfrak{g}$ with a nondegenerate *invariant* scalar product, $(\mathfrak{g}, (\cdot, \cdot))$: $([x, y], z) = (x, [y, z])\ \forall\ x, y, z$; $d = \dim\ \mathfrak{g}$.

Lecture 8: weight systems from Lie algebras

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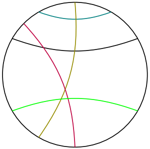
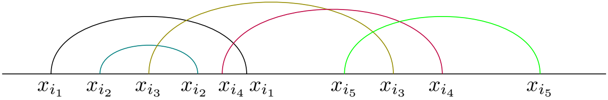
- Pick an orthonormal basis $x_1,\dots,x_d$ in $\mathfrak{g}$, $(x_i,x_j) = \delta_{ij}$.
- Cut the circle of a chord diagram D at some point and make it into an *arc diagram* A.
- Pick a numbering $\nu : V(A) \to \{1,\dots,d\}$ of the arcs of A.
- Put letters $x_{\nu(a)}$ at the ends of each arc a; the result is a word in $U\mathfrak{g}$.
- Sum over all the numberings $\nu : V(A) \to \{1,\dots,d\}$.





$$D \mapsto \sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x_{i_1}x_{i_2}x_{i_3}x_{i_2}x_{i_4}x_{i_1}x_{i_5}x_{i_3}x_{i_4}x_{i_5}$$

Theorem (D. Bar-Natan, M. Kontsevich)

The result is independent of the choice of the orthonormal basis $\{x_i\}$ and the cut point; it belongs to the center of $U\mathfrak{g}$ and satisfies 4-term relations.

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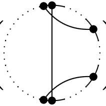
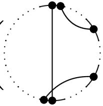
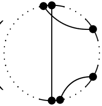
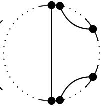
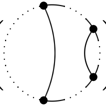
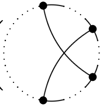
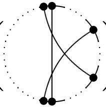
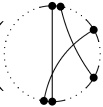
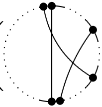
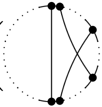
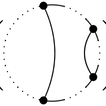
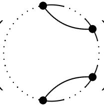
Difficulty: Computations are to be made in a noncommutative algebra.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ is the simplest Lie algebra producing a nontrivial weight system. It consists of 2×2 square matrices with zero trace. The center of its universal enveloping algebra $U\mathfrak{sl}(2)$ is isomorphic to the algebra $\mathbb{C}[c]$ of polynomials in a single variable c, which is the *quadratic Casimir element*. For it, there is a recurrence relation due to Chmutov and Varchenko (1997).

$w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big) + w_{\mathfrak{sl}(2)}\Big($  $\Big)$

$= \; w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big);$

$w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big) + w_{\mathfrak{sl}(2)}\Big($  $\Big)$

$= \; w_{\mathfrak{sl}(2)}\Big($  $\Big) - w_{\mathfrak{sl}(2)}\Big($  $\Big).$

Lecture 8: weight systems from Lie algebras: $\mathfrak{sl}_2$-weight system for complete graphs

The value of the $\mathfrak{sl}(2)$-weight system on a chord diagram depends on the intersection graph of the chord diagram rather than on the diagram itself (S. Chmutov and S. L, 2007).

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The chromatic polynomial for complete graphs on n variables looks very simple: $\chi_{K_n}(c) = c(c-1)\dots(c-n+1) = (c)_n$.
The generating function for it has the continued fraction form

$$\sum_{n=0}^{\infty} \chi_{K_n}(c)t^n = \cfrac{1}{1-ct+\cfrac{ct^2}{1-(c-2)t+\cfrac{(2c-2)t^2}{1-(c-4)t+\cfrac{(3c-6)t^2}{1-(c-6)t+\dots}}}},$$

where the k th row is $1-(c-2(k-1))t+\left(kc-\frac{k(k-1)}{2}\right)t^2$.

Lecture 8: weight systems from Lie algebras: $\mathfrak{sl}_2$-weight system for complete graphs

Theorem (P. Zakorko, 2021, former Lando's conjecture)

We have

$$\begin{aligned}\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_n)t^n &= 1+ct+c(c-1)t^2+c(c-1)(c-2)t^3 \\ &\quad +c(c^3-6c^2+13c-7)t^4+\dots \\ &= \cfrac{1}{1-ct+\cfrac{ct^2}{1-(c-2)t+\cfrac{(4c-3)t^2}{1-(c-6)t+\cfrac{(9c-18)t^2}{1-(c-12)t+\dots}}}},\end{aligned}$$

where the k th row is $1-(c-k(k-1))t+\left(k^2c-\frac{k^2(k^2-1)}{4}\right)t^2$.

Compare with the chromatic continued fraction above: the k th row is $1-(c-2(k-1))t+\left(kc-\frac{k(k-1)}{2}\right)t^2$.

Lecture 8: weight systems from Lie algebras: Values of the $\mathfrak{sl}(2)$-weight system on complete bipartite graphs

Theorem (M. Kazarian, P. Zinova)

For the generating functions $G_m(t) = \sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_{m,n})t^n$, *we have*

$$G_m(t) = \frac{c^m + t \cdot \sum_{i=0}^{m-1} s_{i,m} G_i(t)}{1 - \left(c - \frac{m(m+1)}{2}\right) t}$$

with the initial condition

$$G_0(t) = \frac{1}{1-ct}.$$

There is an explicit formula for the coefficients $s_{i,m}$.

If one replaces complete bipartite graphs sequences $K_{m,n}$, $n = 0, 1, 2, \dots$, with the sequences of *joins* (G, n) of a given graph G with discrete graphs on $n = 0, 1, 2, \dots$ vertices, the form of the previous formula remains the same: the generating function for the values of the $\mathfrak{sl}_2$ weight system is

$$\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}((G,n))t^n = \sum_{m=0}^{|V(G)|} \frac{P_m^G(c)}{1-\left(c-\frac{m(m+1)}{2}\right)t},$$

for some sequence of polynomials $P_0^G, P_1^G, \dots$, which depends on G.

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Theorem (P. Zakorko, P. Zinova)

If we replace a graph G with its complement $\overline{G}$, then the polynomials P_k^G remain the same up to a sign: $P_k^{\overline{G}} = (-1)^{|V(G)|-k} P_k^G$.

Here the *complement graph* $\overline{G}$ has the same set of vertices as G, and the complementary set of edges.

Nothing similar to Chmutov–Varchenko recurrence for other Lie algebras!
Kazarian's idea: For the Lie algebra $\mathfrak{gl}(N)$, a recurrence arises if we extend the weight system from chord diagrams to arbitrary permutations.

Lecture 8: weight systems from Lie algebras: Extending $\mathfrak{gl}(N)$-weight system to permutations

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Version of the initial construction: Pick an arbitrary basis $\{x_1, \dots, x_d\}$, not necessarily orthonormal, and write $x_{\nu(a)}$ on the left end of an arc a and the $(\cdot,\cdot)$-dual element $x^*_{\nu(a)}$ on its right end. In the previous example,

$$\sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x_{i_1}x_{i_2}x_{i_3}x^*_{i_2}x_{i_4}x^*_{i_1}x_{i_5}x^*_{i_3}x^*_{i_4}x^*_{i_5}.$$

The resulting element of the center of the universal enveloping algebra of $\mathfrak{g}$ coincides with the one above.

Lecture 8: weight systems from Lie algebras: Main construction for $\mathfrak{gl}(N)$

For $\mathfrak{g} = \mathfrak{gl}(N)$, with the scalar product $(A, B) := Tr\ AB$, choose the basis consisting of matrix units E_{ij}, $i, j = 1, \dots, N$, with the duality $E_{ij}^* = E_{ji}$.

Definition

For $\sigma \in S_m$, a permutation of m elements, define

$$w_{\mathfrak{gl}(N)} : \sigma \mapsto \sum_{i_1, i_2, \dots, i_m = 1}^{N} E_{i_1, i_{\sigma(1)}} E_{i_2, i_{\sigma(2)}} \dots E_{i_m, i_{\sigma(m)}} \in U\mathfrak{gl}(N).$$

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Theorem

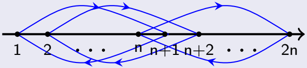
For any permutation σ, $w_{\mathfrak{gl}(N)}(\sigma)$ lies in the center $ZU\mathfrak{gl}(N)$ of $U\mathfrak{gl}(N)$.

Lecture 8: weight systems from Lie algebras: Graph of a permutation

Definition (digraph of the permutation)

A permutation can be represented as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are placed on the horizontal line, and numbered from left to right in the increasing order. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges, for example:

$$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$$



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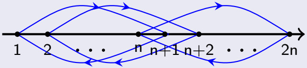

Chord diagrams are permutations of special kind, involutions without fixed points. For them, the initial definition coincides with the one above.

Lecture 8: weight systems from Lie algebras: The center $ZU\mathfrak{gl}(N)$

Define *Casimir elements* $C_m \in U\mathfrak{gl}(N)$, $m = 1, 2, \dots$:

$$C_m = w_{\mathfrak{gl}(N)}((1, 2, \dots, m)) = \sum_{i_1, i_2, \dots, i_m = 1}^{N} E_{i_1, i_2} E_{i_2, i_3} \dots E_{i_m, i_1};$$

associated to the standard cycles $1 \mapsto 2 \mapsto 3 \mapsto \dots \mapsto m \mapsto 1$.

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Theorem

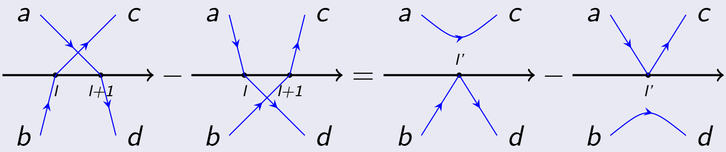
The center $ZU\mathfrak{gl}(N)$ of the universal enveloping algebra $U\mathfrak{gl}(N)$ of $\mathfrak{gl}(N)$ is identified with the polynomial ring $\mathbb{C}[C_1, \dots, C_N]$.

Lecture 8: weight systems from Lie algebras: Recurrence relation

Theorem (Zhuoke Yang)

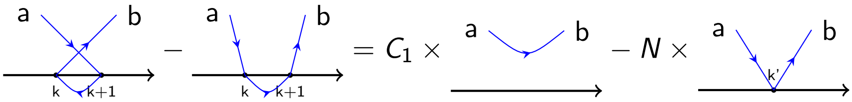
The $w_{\mathfrak{gl}(N)}$ invariant of permutations possesses the following properties:

- *for the empty permutation, the value of $w_{\mathfrak{gl}(N)}$ is equal to 1;*
- *$w_{\mathfrak{gl}(N)}$ is multiplicative with respect to concatenation of permutations;*
- (**Recurrence Rule**) *For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $l, l+1$, of the permuted set $\{1, 2, \dots, m\}$, we have for the values of the $w_{\mathfrak{gl}(N)}$ weight system*



Lecture 8: weight systems from Lie algebras: Recurrence rule for the special case

For the special case $\sigma(k+1) = k$, the recurrence looks like follows:



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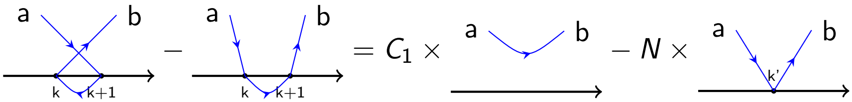
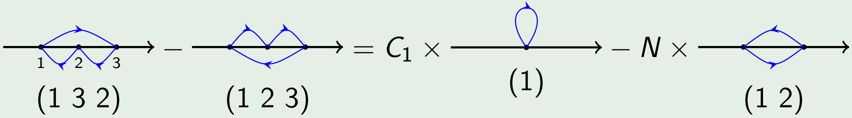


Example



$$
\begin{aligned}
w_{\mathfrak{gl}(N)}((1\ 3\ 2)) &= w_{\mathfrak{gl}(N)}((1\ 2\ 3)) + C_1 \cdot w_{\mathfrak{gl}(N)}((1)) - N \cdot w_{\mathfrak{gl}(N)}((1\ 2)) \\
&= C_3 + C_1^2 - NC_2
\end{aligned}
$$

Lecture 8: weight systems from Lie algebras: Universal $\mathfrak{gl}$-weight system

Corollary

The $\mathfrak{gl}(N)$-weight systems, for $N = 1, 2, \ldots$, are combined into a universal $\mathfrak{gl}$-weight system $w_{\mathfrak{gl}}$ taking values in the ring of polynomials in infinitely many variables $\mathbb{C}[N; C_1, C_2, \ldots]$.
After substituting a given value of N and an expression of higher Casimirs $C_{N+1}, C_{N+2}, \ldots$ in terms of the lower ones $C_1, C_2, \ldots, C_N$, this weight system specifies into the $\mathfrak{gl}(N)$-weight system.

Lecture 8: weight systems from Lie algebras: Inducing graph invariants from the universal $\mathfrak{gl}$-weight system

For the chord diagram of order 5 such that any two chords intersect one another, we have

$$\begin{aligned} w_{\mathfrak{gl}}(K_5) \;=\; & 24C_2N^4+(24C_3-50C_2^2-24C_1^2)N^3 \\ & -(24C_4+10C_2C_3-35C_2^3-70C_1^2C_2+72C_1C_2-32C_2)N^2 \\ & +(10C_2C_4+96C_1C_3-10C_2^4-50C_1^2C_2^2+30C_1C_2^2-82C_2^2-20C_1^4+48C_1^3-32C_1^2)N \\ & -40C_1C_2C_3+C_2^5+10C_1^2C_2^3+30C_2^3+15C_1^4C_2-20C_1^3C_2+10C_1^2C_2, \end{aligned}$$

What kind of information does this polynomial contain?

Lecture 8: weight systems from Lie algebras: Inducing graph invariants from the universal $\mathfrak{gl}$-weight system

Theorem (N. Kodaneva, S. L., 2023)

The composition of the following substitutions for N and C_k, $k = 1, 2, 3, \ldots$ makes the value of $w_{\mathfrak{gl}}$ on a chord diagram into the interlace polynomial of its intersection graph in the variable $z^2 - 1$:

$$C_k = \frac{1}{N}((N+\epsilon)^k - (1-N^2)\epsilon^k); \qquad N = z^2 - 1, \epsilon = \frac{1}{1-z}.$$

In particular, for K_5 the substitution gives the polynomial $16z^2$, which is the interlace polynomial $Q_{K_5}(z^2 - 1)$.

It is easy to show that no substitution for $N, C_1, C_2, \ldots$ makes $w_{\mathfrak{gl}}$ into the chromatic polynomial of the intersection graph of a chord diagram: the corresponding system of equations for K_1, K_2, K_3, K_4, K_5 has no solutions.

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Theorem

Under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \dots$, the value of $w_{\mathfrak{gl}}$ on a chord diagram becomes a polynomial in N whose leading term is the chromatic polynomial of the intersection graph of the chord diagram.

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Theorem

The assertion remains true if one replaces chord diagram with an arbitrary positive permutation.

A permutation is *positive* if each of its disjoint cycles is strictly increasing, with the exception of the last element.

Lecture 8: weight systems from Lie algebras: An extension for Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{so}$, $\mathfrak{sp}$

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There is an extension of the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system to permutations similar to that for the Lie algebra $\mathfrak{gl}(N)$. The corresponding universal weight system, which works for all values of m and n together, coincides with the result of substitution $N = m - n$ into the universal weight system $w_{\mathfrak{gl}}$.

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For the other classical series of Lie algebras and Lie superalgebras, the corresponding construction is elaborated by M.Kazarian and Zhoke Yang.

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Lecture 8: weight systems from Lie algebras: Open problems

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- Same questions about interlace polynomial.

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Lecture 8: problems

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Thank you
for your attention