Invariants of graphs, embedded graphs, delta-matroids, and permutations

Sergei Lando

National Research University Higher School of Economics,
Skolkovo Institute of Science and Technology,
Moscow, Russia

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Lecture 7: 4-term relation for graphs: Definition

To each chord diagram, a graph, its intersection graph. is associated. Hence, any graph invariant determines a function on chord diagrams. It is interesting to know, which of these functions are weight systems and lead, therefore, to knot invariants.

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Definition

A graph invariant f *satisfies* 4-*term relations* if for any graph G and any pair A, B of its vertices we have

$$f(G) - f(G'_{AB}) = f(\widetilde{G}_{AB}) - f(\widetilde{G'}_{AB}),$$

where G'_{AB} is obtained from G by switching the adjacency between A and B, $\widetilde{G}_{AB}$ is the result of switching the adjacency to A of all vertices adjacent to B.

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Theorem

Any graph invariant satisfying 4-*term relations determines a weight system.*

Lecture 7: 4-term relation for graphs: Examples

Example

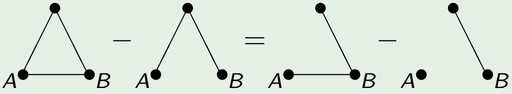

Figure: A 4-term relation for graphs with 3 vertices

Lecture 7: 4-term relation for graphs: Examples

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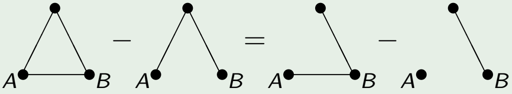

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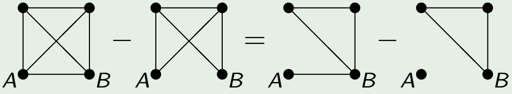

Figure: A 4-term relation for graphs with 4 vertices

Example

The corank of the adjacency matrix $A(G)$ of a graph G satisfies 4-term relations. In fact, the corank of the adjacency matrix of the intersection graph of a chord diagram, when increased by 1, yields the number of faces of the chord diagram in question.

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Example

The chromatic polynomial of a graph G satisfies 4-term relations.

In 1995, R. Stanley generalized the chromatic polynomial, which is a polynomial in one variable, to a graph invariant taking values in the ring of polynomials in infinitely many variables. Let $h : V(G) \to X = \{x_1, x_2, x_3, \dots\}$ be a mapping. We associate to this mapping the monomial m_h, which is the product of the values of h on the vertices, $m_h = \prod_{v \in V(G)} h(v)$.

Then *Stanley's symmetrized chromatic polynomial* $S_G(x_1, x_2, \dots)$ is defined as

$$S_G(x_1, x_2, \dots) = \sum_{\substack{h: V(G) \to X \\ h \text{ proper}}} m_h.$$

Here a coloring h is *proper* if any two neighboring vertices v_1, v_2 are colored differently, $h(v_1) \neq h(v_2)$.

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Stanley's symmetrized chromatic polynomial is equivalent to the weighted chromatic polynomial, introduced by S. Chmutov, S. Duzhin and S. L. in 1994.

A *weighted graph* is a simple graph G together with a *weight*, which associates a positive integer to each vertex. Define the *weighted chromatic polynomial* $W(p_1, p_2, p_3, \dots)$ by the relations:

- the weighted chromatic polynomial takes an isolated vertex of weight n to p_n;
- it is multiplicative, $W_{G_1 \sqcup G_2} = W_{G_1} W_{G_2}$;
- it satisfies the *deletion-contraction relation*

$$W_G = W_{G'_e} + W_{G''_e}$$

  for an arbitrary edge $e \in E(G)$, where G'_e is the result of deleting G, and G''_e is the result of contracting e; the new vertex which arises under contraction gets the weight equal to the sum of the weights of two ends of e.

Theorem

Make a simple graph into a weighted graph by assigning weight 1 to each vertex. The weighted chromatic polynomial of these weighted graphs satisfies 4-term relations for graphs.

Lecture 7: 4-term relation for graphs: 4-term relations for delta-matroids

Theorem

The interlace polynomial of a graph satisfies 4-term relations.

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Theorem

The interlace polynomial of a graph satisfies 4-term relations.

Maybe the easiest way to prove this assertion is to introduce 4-term relations for delta-matroids, and then prove that the interlace polynomial of delta-matroids satisfies these 4-term relations.
The 4-term relations for delta-matroids have the same form

$$f(D) - f(D'_{ab}) = f(\widetilde{D_{ab}}) - f(\widetilde{D'_{ab}})$$

as for chord diagrams or graphs, but we need to introduce two operations $D \mapsto D'_{ab}$ (the first Vassiliev move) and $D \mapsto \widetilde{D_{ab}}$ (the second Vassiliev move).

Lecture 7: 4-term relation for graphs: Vassiliev moves for delta-matroids

It is easier to define the second Vassiliev move:

$\widetilde{D_{ab}} = (\mathcal{E}; \widetilde{\Phi_{ab}}),$ where $\widetilde{\Phi_{ab}} = \Phi\Delta\{\phi\sqcup\{a\} | \phi\sqcup\{b\} \in \Phi$ and $\phi \subset \mathcal{E}\setminus\{a, b\}\}.$

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The first Vassiliev move is then defined as follows:

$$D'_{ab} = (\mathcal{E}; \Phi'_{ab}), \text{ where } \Phi'_{ab} = \widetilde{(\Phi * b)}_{ab} * b.$$

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Theorem

For delta-matroids of embedded graphs, the first and the second Vassiliev moves are consistent with the ordinary operations.

- A *perfect matching* in a graph G is a subset of the set $E(G)$ of its vertices such that each vertex of G belongs to a single edge in the subset. For example, the number of perfect matchings in the complete graph K_n is 0 if n is odd and is $(2m-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)$ provided $n = 2m$ is even. Prove that the number of perfect matchings satisfies 4-term relations for graphs.
- Extend the previous result to the graph *matching polynomial* defined as follows:

$$M_G(t) = \sum_F t^{|F|},$$

where the sum runs over all subsets $F \subset E(G)$ such that each vertex in $V(G)$ is an end of at most one edge in F.

- Prove that the substitution $p_i = (-1)^i c$, $i = 1, 2, \ldots$ makes Stanley's symmetrized chromatic polynomial into the chromatic polynomial.
- Prove that Stanley's symmetrized chromatic polynomial satisfies 4-term relations for graphs.
- Prove that the interlace polynomial satisfies 4-term relations for delta-matroids.

Thank you
for your attention