Invariants of graphs, embedded graphs, delta-matroids, and permutations

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Lecture 2: Embedded graphs

1. Two-dimensional surfaces
2. Orientability
3. Graphs on surfaces
4. Duality
5. Permutation presentation of a graph on a surface
6. Number of faces

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Lecture 2: Embedded graphs: Two-dimensional surfaces

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A surface is *connected* provided any two points on it can be connected by a continuous path.

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Below, we will be mainly interested in connected and orientable surfaces, addressing to other types of surfaces only occasionally.

As we glue a surface from polygons, the edges of the polygons form a graph on a surface (embedded graph). The vertices of the graph are the vertices of the polygons (at each vertex of the graph one or more vertices of the polygons are glued together). The edges of the graph are the edges of the polygons (at each edge of the graph exactly two edges of one or two different polygons are glued together).

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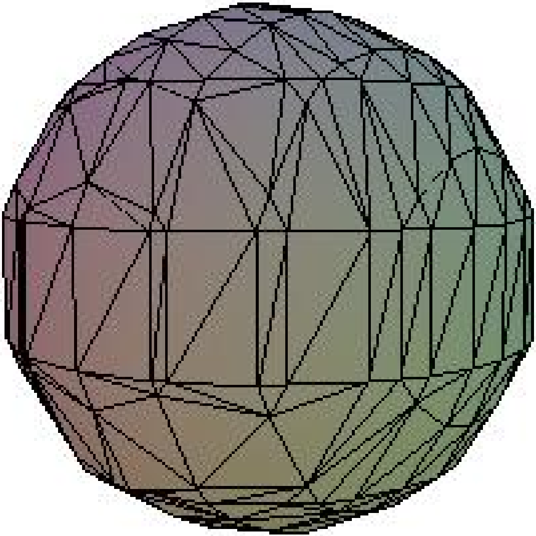
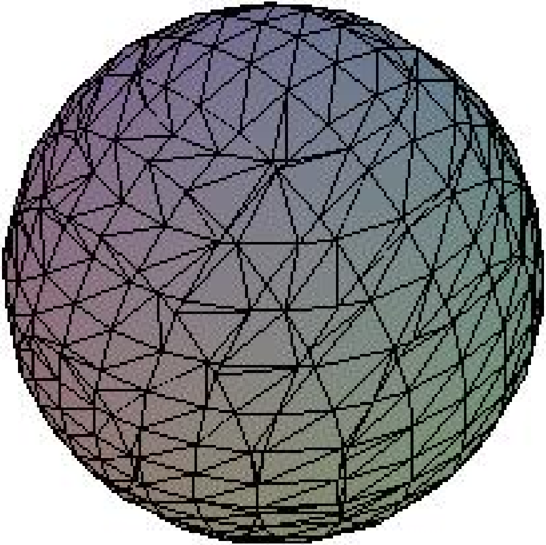
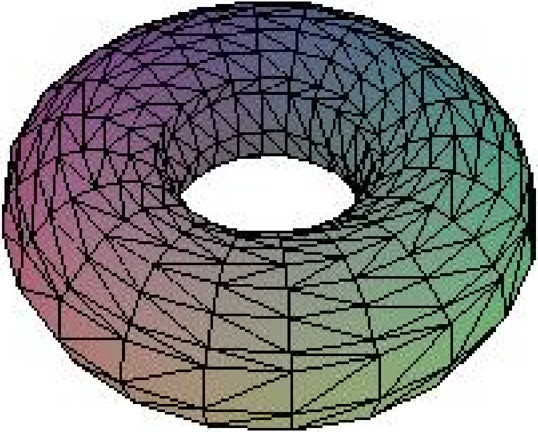
A graph is said to be *planar* if it is embeddable into the sphere.







Macbeath surface

To each embedded graph Γ its *dual* embedded graph $\overline{\Gamma}$ is associated. This is a graph on the same surface as Γ whose vertices are in one-to-one correspondence with the faces $F(\Gamma)$ of Γ, edges are in one-to-one correspondence with that of Γ, and faces are in one-to-one correspondence with the vertices $V(\Gamma)$ of Γ.

Lecture 2: Embedded graphs: Duality

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The dual graph $\overline{\Gamma}$ can be drawn as follows. Pick a point inside each face of Γ (a 'center'). These points will be the vertices of $\overline{\Gamma}$. Each edge of Γ determines an edge in $\overline{\Gamma}$: this edge connects the centers of the two faces incident to the original edge; it does not intersect any edge of Γ but the original one. (Note: the two faces incident to a given edge may be one and the same face.) Finally, the faces of $\overline{\Gamma}$ are the polygons in which the union of the new edges cut the surface.

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In order to describe the second, and in many senses more convenient, presentation remark first that the embedding of a graph into an oriented surface specifies a *cyclic order* of half-edges around each vertex. A *half-edge* (or a *flag*) consists of a vertex and an edge incident to it. The cyclic order takes each half-edge to the next one in the positive direction (with respect to orientation of the surface) around the corresponding vertex.

Theorem

For a given graph, specifying a cyclic order of half-edges around each vertex determines the corresponding embedded graph uniquely.

Let H be a set consisting of even number $2n$ of elements. An *embedded graph* (or a *combinatorial map*) on H is a triple of permutations $(\sigma, \alpha, \varphi)$ of H such that

- α is an involution without fixed points;
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σ specifies rotation of half-edges around vertices; it identifies vertices as orbits of σ.
α identifies edges as orbits of α: each edge consists of two half-edges forming the orbit.
φ identifies faces as orbits of φ: it takes a half-edge to the one at the next vertex of the face, which looks in the same direction.

Replacing the triple of permutations $(\sigma, \alpha, \varphi)$ by the triple $(\varphi^{-1}, \alpha^{-1}, \sigma^{-1})$ makes an embedded graph into the dual one.
Note that since α is an involution without fixed points, we have $\alpha^{-1} = \alpha$.

For a given connected graph embedded into an orientable surface, the numbers V, E, F of vertices, edges and faces, determine the topology of the surface uniquely. Namely, two surfaces have the same topology (are 'homeomorphic') iff the numbers $V - E + F$ coincide. This common number is called the *Euler characteristic* of the surface (or of the embedded graph).
In terms of permutations, V, E, and F are the numbers of disjoint cycles in the corresponding permutations,

$$V = c(\sigma), \qquad E = c(\alpha), \qquad F = c(\varphi).$$

Dual embedded graphs are embedded in the same surface, whence have the same Euler characteristic,

$$V - E + F = F - E + V.$$

Euler characteristic of an orientable surface is even. It equals $2 - 2g$, where g is the number of *handles* attached to the sphere necessary to obtain a given surface.

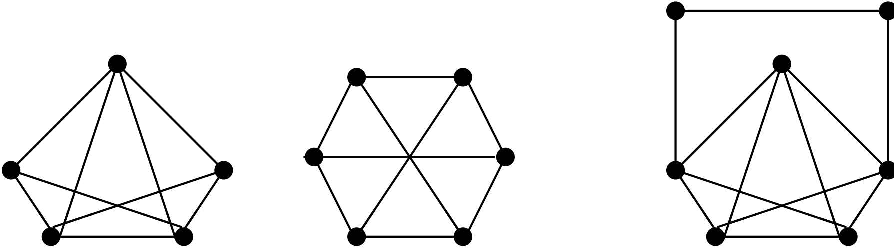

Figure: Which of these graphs are planar?

Lecture 2: Embedded graphs: Problems

- What is the maximal genus of the surface that can be obtained by gluing the edges of a $2n$-gon?
- How many are there ways to glue the edges of an octagon in pairs? How many of these gluings produce a) the sphere; b) the torus; c) the pretzel; d) surface of genus 3?
- What is the genus of a surface obtained by gluing in pairs the opposite edges of a $4n$-gon?
- What is the dual graph to the one in the previous problem?

- Prove that the graphs K_5 and $K_{3,3}$ are not planar.
- For each of the graphs $K_3, K_4, K_5, K_6, K_7, K_8, K_{2,3}, K_{3,3}, K_{3,4}$ either construct its embedding into the torus or prove that there is no such embedding.
- An embedded graph is said to be *autodual* if its dual embedded graph is isomorphic to it. For each n, construct an autodual embedded graph with n vertices on the sphere.
- How many edges does have an autodual embedded graph with n vertices a) on the torus? b) on a genus g surface?
- Draw the embedded graphs presented by permutations (written in the cyclic form) a) $\alpha = (1,2)(3,4)(5,6)$, $\sigma = (1,3,5,6)(2)(4)$;
b) $\alpha = (1,2)(3,4)(5,6)$, $\sigma = (1,5,3,6)(2)(4)$).

Lecture 2: Embedded graphs: Problems

- The cube is usually considered as a map of genus 0 having 6 faces of degree 4. Embed the graph of the cube into a torus in such a way that the corresponding map would have 4 faces of degree 6.
- What will be the dual graph to the cube in this embedding?
- The icosahedron is usually considered as a map of genus 0 having 12 vertices of degree 5, 30 edges, and 20 triangular faces. Embed the graph of the icosahedron into a surface of genus 4 in such a way that the corresponding map would have 12 faces of degree 5.
- What will be the dual graph to the icosahedron in this embedding?

Thank you for your attention