Invariants of graphs, embedded graphs, delta-matroids, and permutations

Sergei Lando

National Research University Higher School of Economics,
Skolkovo Institute of Science and Technology,
Moscow, Russia

August 11–25, 2024

References

Books:

S.Chmutov, S.Duzhin, and J.Mostovoy, *Introduction to Vassiliev Knot Invariants*, Cambridge University Press (2012)
J.A. Ellis-Monaghan, I.Moffatt, *Graphs on surfaces: dualities, polynomials, and knots*, Springer, 2013/6/28 84
J.A. Ellis-Monaghan, I.Moffatt, *Handbook of the Tutte polynomial and related topics*, CRC Press (2022)
S.Lando and A.Zvonkin, *Graphs on Surfaces and Their Applications*, Springer (2004)

Lecture 1: Graphs and their invariants

1. Families of graphs
2. Graph presentations
3. Graph isomorphism
4. Examples of graph invariants: chromatic function, degeneracy
5. Deletion-contraction relation

A *simple graph* G is a pair $(V(G), E(G))$ consisting of a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges*, which are unordered pairs of distinct vertices.
The two vertices forming an edge are called its *ends*.
The *degree* of a vertex is the number of edges for which it is an end.

A *simple graph* G is a pair $(V(G), E(G))$ consisting of a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges*, which are unordered pairs of distinct vertices.
The two vertices forming an edge are called its *ends*.
The *degree* of a vertex is the number of edges for which it is an end.

A *simple path* of length ℓ in G is a sequence of pairwise distinct vertices $v_0, v_1, \dots, v_\ell$, $v_i \in V(G)$, such that each pair of consecutive vertices v_i, v_{i+1} is an edge. A *simple circuit* of length ℓ is defined similarly, but with the requirement $v_\ell = v_0$.
A graph is said to be *connected* if any two vertices in it are connected by a path.

Graphs are usually drawn on the plane, with vertices shown as points, and edges with line or curve segments:

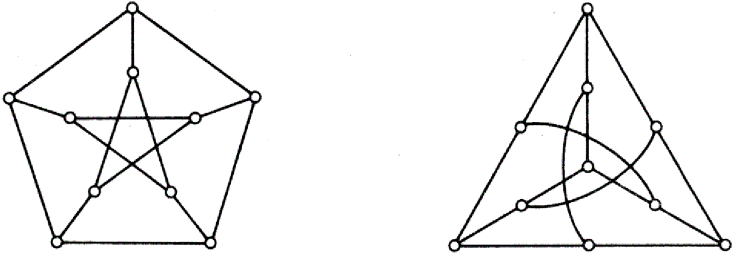

Figure: Pictures of two graphs

Graphs having similar structure form graph families. We will widely use the following families of graphs:

Graphs having similar structure form graph families. We will widely use the following families of graphs:

- a graph is a *tree* if it is connected and has no circuits; if a graph has no circuits, but is not necessarily connected, then it is a *forest*;

Graphs having similar structure form graph families. We will widely use the following families of graphs:

- a graph is a *tree* if it is connected and has no circuits; if a graph has no circuits, but is not necessarily connected, then it is a *forest*;
- the *cycle* C_n is the graph consisting of a single circuit, of length n;

Graphs having similar structure form graph families. We will widely use the following families of graphs:

- a graph is a *tree* if it is connected and has no circuits; if a graph has no circuits, but is not necessarily connected, then it is a *forest*;
- the *cycle* C_n is the graph consisting of a single circuit, of length n;
- a graph is *complete* if any pair of vertices in it is connected by an edge; the complete graph on n vertices is denoted K_n;

Graphs having similar structure form graph families. We will widely use the following families of graphs:

- a graph is a *tree* if it is connected and has no circuits; if a graph has no circuits, but is not necessarily connected, then it is a *forest*;
- the *cycle* C_n is the graph consisting of a single circuit, of length n;
- a graph is *complete* if any pair of vertices in it is connected by an edge; the complete graph on n vertices is denoted K_n;
- a graph G is said to be *bipartite* if there is a splitting $V(G) = U_1 \sqcup U_2$ of the set of its vertices into two disjoint parts such that each edge connects a vertex from U_1 to a vertex in U_2;

Graphs having similar structure form graph families. We will widely use the following families of graphs:

- a graph is a *tree* if it is connected and has no circuits; if a graph has no circuits, but is not necessarily connected, then it is a *forest*;
- the *cycle* C_n is the graph consisting of a single circuit, of length n;
- a graph is *complete* if any pair of vertices in it is connected by an edge; the complete graph on n vertices is denoted K_n;
- a graph G is said to be *bipartite* if there is a splitting $V(G) = U_1 \sqcup U_2$ of the set of its vertices into two disjoint parts such that each edge connects a vertex from U_1 to a vertex in U_2;
- a bipartite graph G is *complete bipartite* if each vertex in U_1 is connected to each vertex in U_2; the complete bipartite graph with parts of size m, n is denoted $K_{m,n}$.

A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.
For a graph $G = (V(G), E(G))$ and a subset of its vertices $I \subset V(G)$, the subgraph G_I of G *induced* by I is the subgraph $(I, E(G_I))$, whose set of edges $E(G_I)$ consists of all edges in $E(G)$ whose both ends belong to I.

A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.
For a graph $G = (V(G), E(G))$ and a subset of its vertices $I \subset V(G)$, the subgraph G_I of G *induced* by I is the subgraph $(I, E(G_I))$, whose set of edges $E(G_I)$ consists of all edges in $E(G)$ whose both ends belong to I.
Obvious assertions:

- any subgraph of a tree is a forest;
- any induced subgraph of a complete graph is a complete graph;
- any subgraph of a bipartite graph is a bipartite graph;
- any induced subgraph of a complete bipartite graph is a complete bipartite graph.

A graph $H = (V(H), E(H))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.
For a graph $G = (V(G), E(G))$ and a subset of its vertices $I \subset V(G)$, the subgraph G_I of G *induced* by I is the subgraph $(I, E(G_I))$, whose set of edges $E(G_I)$ consists of all edges in $E(G)$ whose both ends belong to I.
Obvious assertions:

- any subgraph of a tree is a forest;
- any induced subgraph of a complete graph is a complete graph;
- any subgraph of a bipartite graph is a bipartite graph;
- any induced subgraph of a complete bipartite graph is a complete bipartite graph.

In this sense, the families we are considering are *hereditary*.

Lecture 1: Graphs and their invariants: Graphs' presentations

The most natural way to represent a graph G in a computer-friendly form is to specify its set of vertices $V(G)$ and its set of edges $E(G)$, which is a subset of the direct square $V(G) \times V(G)$.

Lecture 1: Graphs and their invariants: Graphs' presentations

The most natural way to represent a graph G in a computer-friendly form is to specify its set of vertices $V(G)$ and its set of edges $E(G)$, which is a subset of the direct square $V(G) \times V(G)$.
Another natural way is to number the vertices by the numbers $1, 2, \dots,$ $|V(G)|$ and represent G by means of the *adjacency matrix* $A(G)$, which is a symmetric square $|V(G)| \times |V(G)|$-matrix whose entry a_{kl}, $1 \le k, l \le |V(G)|$, is 1 provided the vertices numbered k and l are connected by an edge, and is 0 otherwise. In particular, the diagonal elements of the adjacency matrix $A(G)$ of a simple graph G are zeroes.

Lecture 1: Graphs and their invariants: Graphs' presentations

The most natural way to represent a graph G in a computer-friendly form is to specify its set of vertices $V(G)$ and its set of edges $E(G)$, which is a subset of the direct square $V(G) \times V(G)$.
Another natural way is to number the vertices by the numbers $1, 2, \dots,$ $|V(G)|$ and represent G by means of the *adjacency matrix* $A(G)$, which is a symmetric square $|V(G)| \times |V(G)|$-matrix whose entry a_{kl}, $1 \leq k, l \leq |V(G)|$, is 1 provided the vertices numbered k and l are connected by an edge, and is 0 otherwise. In particular, the diagonal elements of the adjacency matrix $A(G)$ of a simple graph G are zeroes.
One more way to encode a graph G is by means of its *incidence matrix* $I(G)$. This is a rectangular $|V(G)| \times |E(G)|$-matrix with entries 0 and 1, whose entry i_{kl}, $1 \leq k \leq |V(G)|$, $1 \leq l \leq |E(G)|$, is 1 if vertex number k is an end of the edge number l and is 0 otherwise. To construct an incidence matrix, one should number both vertices and edges of G.

Two graphs G_1, G_2 are said to be *isomorphic* if there is a one-to-one correspondence $\varphi : V(G_1) \to V(G_2)$ taking the edges of G_1 to that of G_2 such that the inverse map φ^{-1} takes the edges of G_2 to that of G_1. Such φ is called an *isomorphism* between G_1 and G_2.

Two graphs G_1, G_2 are said to be *isomorphic* if there is a one-to-one correspondence $\varphi : V(G_1) \to V(G_2)$ taking the edges of G_1 to that of G_2 such that the inverse map φ^{-1} takes the edges of G_2 to that of G_1. Such φ is called an *isomorphism* between G_1 and G_2.
Finding whether two given graphs are isomorphic is one of the key problems in studying graphs. **Caution:** Exhausting all one-to-one mappings $V(G_1) \to V(G_2)$ and checking for each of them whether it is an isomorphism is impractical already for graphs with as few as 30 vertices. Presently, it is not known whether the graph isomorphism problem can be solved by means of a polynomial algorithm.

A function on graphs is called a *graph invariant* if it takes the same values on any pair of isomorphic graphs.
Graph invariants are a useful tool to proving that two given graphs are *not* isomorphic. They are also extensively used in studying graph properties.

Examples of graph invariants:

- number of vertices;
- number of edges;
- automorphism group;
- chromatic number (minimal number of colors required to color the vertices of a graph properly);
- characteristic polynomial of the adjacency matrix;
- length of the longest circuit;
- length of the shortest circuit;
- ...

Examples of graph invariants:

- number of vertices;
- number of edges;
- automorphism group;
- chromatic number (minimal number of colors required to color the vertices of a graph properly);
- characteristic polynomial of the adjacency matrix;
- length of the longest circuit;
- length of the shortest circuit;
- ...

A graph invariant is good if it

- can be computed relatively easily;
- separates graphs well.

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \ldots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \ldots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)

Obviously, the chromatic function is multiplicative: its value on the disjoint union $G_1 \sqcup G_2$ of graphs is the product of its values on the components, $\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$. Therefore, it suffices to know the value of χ on connected graphs.

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \ldots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)

Obviously, the chromatic function is multiplicative: its value on the disjoint union $G_1 \sqcup G_2$ of graphs is the product of its values on the components, $\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$. Therefore, it suffices to know the value of χ on connected graphs.

The chromatic function on certain families of connected graphs:

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \dots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)
Obviously, the chromatic function is multiplicative: its value on the disjoint union $G_1 \sqcup G_2$ of graphs is the product of its values on the components, $\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$. Therefore, it suffices to know the value of χ on connected graphs.
The chromatic function on certain families of connected graphs:

- if a graph G has a *leaf* (a vertex of degree 1), then let G' denote the result of erasing the leaf from G; we have $\chi_G(c) = (c-1)\chi_{G'}(c)$;

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \ldots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)
Obviously, the chromatic function is multiplicative: its value on the disjoint union $G_1 \sqcup G_2$ of graphs is the product of its values on the components, $\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$. Therefore, it suffices to know the value of χ on connected graphs.
The chromatic function on certain families of connected graphs:

- if a graph G has a *leaf* (a vertex of degree 1), then let G' denote the result of erasing the leaf from G; we have $\chi_G(c) = (c-1)\chi_{G'}(c)$;
- the chromatic function of any tree T_n on n vertices is $\chi_{T_n}(c) = c(c-1)^{n-1}$;

Let $\chi_G(c)$ denote the graph invariant that counts the number of proper colorings of the vertices of G into c colors. (A coloring $V(G) \to \{1, 2, \dots, c\}$ is *proper* if any two neighboring vertices of the graph are taken to different colors.)

Obviously, the chromatic function is multiplicative: its value on the disjoint union $G_1 \sqcup G_2$ of graphs is the product of its values on the components, $\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$. Therefore, it suffices to know the value of χ on connected graphs.

The chromatic function on certain families of connected graphs:

- if a graph G has a *leaf* (a vertex of degree 1), then let G' denote the result of erasing the leaf from G; we have $\chi_G(c) = (c-1)\chi_{G'}(c)$;
- the chromatic function of any tree T_n on n vertices is $\chi_{T_n}(c) = c(c-1)^{n-1}$;
- the chromatic function of the complete graph K_n on n vertices is $\chi_{K_n}(c) = c(c-1)(c-2)\dots(c-n+1)$.

A more complicated task is to compute the chromatic function of the cycle C_4 on 4 vertices.

A more complicated task is to compute the chromatic function of the cycle C_4 on 4 vertices.
Pick an edge $e \in E(C_4)$. Then we have

$$\chi_{C_4}(c) = \chi_{(C_4)'_e}(c) - \chi_{(C_4)''_e}(c),$$

where $(C_4)'_e$ is the result of deleting e and $(C_4)''_e$ is the result of contracting e in C_4.

A more complicated task is to compute the chromatic function of the cycle C_4 on 4 vertices.
Pick an edge $e \in E(C_4)$. Then we have

$$\chi_{C_4}(c) = \chi_{(C_4)'_e}(c) - \chi_{(C_4)''_e}(c),$$

where $(C_4)'_e$ is the result of deleting e and $(C_4)''_e$ is the result of contracting e in C_4. Indeed, all proper colorings of the vertices of the graph $(C_4)'_e$ split into two subsets: those in which the ends of e are colored into two different colors, and those in which these two colors coincide. The first subset is in one-to-one correspondence with the proper colorings of C_4, while the second subset corresponds one-to-one with the set of proper colorings of the graph $(C_4)''_e$. Since $(C_4)'_e$ is a tree and $(C_4)''_e$ is $C_3 = K_3$, we already know the corresponding chromatic functions.

Theorem

Let G be a graph, and let $e \in E(G)$ be an arbitrary edge. Then

$$\chi_G(c) = \chi_{G'_e}(c) - \chi_{G''_e}(c),$$

where G'_e is the result of erasing e, G''_e is the result of contracting e in G.

Theorem

Let G be a graph, and let $e \in E(G)$ be an arbitrary edge. Then

$$\chi_G(c) = \chi_{G'_e}(c) - \chi_{G''_e}(c),$$

where G'_e is the result of erasing e, G''_e is the result of contracting e in G.

Corollary

For any graph G, the function $\chi_G(c)$ is a polynomial in c of degree $|V(G)|$.

Lecture 1: Graphs and their invariants: Problems

- Prove that if a graph G with $|V(G)|$ vertices has more than

$$\binom{|V(G)|-1}{2} = \frac{1}{2}(|V(G)|-1)(|V(G)|-2)$$

  edges, then it is connected.
- Prove that a graph is bipartite if and only if any circuit in it has an even length.
- Let $I(G)$ denote the incidence matrix of a graph G. Prove that the diagonal elements of the $|V(G)| \times |V(G)|$ matrix $I(G)I(G)^t$, where $I(G)^t$ is the transpose to $I(G)$, are the degrees of the vertices of G.

- Compute the chromatic polynomial of the graphs a) $K_{2,3}$, complete bipartite graph with parts of size 2 and 3; a) $K_{3,3}$, complete bipartite graph with parts of size 3 and 3;
- Find a formula for the chromatic polynomial of the cycle C_n, $n = 3, 4, 5, \dots;$
- Prove that the chromatic polynomial possesses the following *binomial property*:

$$\chi_G(x+y) = \sum_{I \sqcup J = V(G)} \chi_{G|_I}(x)\chi_{G|_J}(y),$$

where the summation is carried over all partitions of the set $V(G)$ of vertices of G into two disjoint subsets I, J. Recall that $G|_I$ is the subgraph of G induced by the subset $I \subset V(G)$.

Thank you
for your attention