# Multispreads and additive intriguing sets in Hamming graphs

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# Introduction

spreads $\subset$ $\mu$-fold spreads $\subset$ **multispreads** $\subset$ $\mu$-fold space partitions

- ($\mu$-fold) spreads, or $t$-spreads — (multifold) partitions of the vector space into $t$-subspaces;
- **multispreads** — $\mu$-fold partitions of the vector space $\mathbb{F}_q^m$ into $t$-multisubspaces (subspaces of different dimensions $r \le t$, counted with multiplisity $q^{t-r}$);
- (multifold) space partitions — (multifold) partitions of the vector space into subspaces of (possibly) different dimensions;
- Motivation: correspondence between multispreads, additive *intriguing sets* (completely regular codes with covering radius 1) in Hamming graphs, and additive one-weight codes.

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# OUTLINE

- definitions;
- connection: multispreads, intriguing sets (CR-1), one-weight codes;
- multispreads: necessary condition;
- multispreads: simple special cases;
- multispreads: constructions;
- multispreads: characterized cases:
  corresponding to additive intriguing sets in
  $H(n,p^2)$, $H(n,2^3)$, $H(n,2^4)$, $H(n,3^3)$;
- duality

# Definition: intriguing sets

- A set of vertices of a regular (Hamming) graph is called an intriguing set (completely regular code of covering radius $\mathbf{1}$, CR-$\mathbf{1}$) with quotient matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  if every codeword is adjacent to $a$ codewords and $b$ non-codewords and every non-codeword is adjacent to $c$ codewords and $d$ non-codewords.
- equivalent notions: equitable 2-partitions, perfect 2-colorings, 2-partition designs, . . .

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- equivalent notions: equitable 2-partitions, perfect 2-colorings, 2-partition designs, ...

# Notations from coding theory

- $\mathbb{F}_q^n$ — the space of $n$-tuples (words) of elements of $\mathbb{F}_q = \mathrm{GF}(q)$.
- The weight of a word is the number of nonzero elements in it; the Hamming distance between two words is the number of positions in which they differ.
- A subspace (linear code) or an additive subgroup (additive code) of $\mathbb{F}_q^n$ is said to be a one-weight code if all non-zero codewords have the same weight.
- If $q = p^t$, $p$ prime, then additive codes are just $\mathbb{F}_p$-linear codes in $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is considered as $t$-dimensional vector space over $\mathbb{F}_p$.
- Choosing any $\mathbb{F}_p$-basis in $\mathbb{F}_q$, we can represent $\mathbb{F}_q^n$ as $(\mathbb{F}_p^t)^n$, the set of words of length $n$ over $\mathbb{F}_p^t$,
  or the set of words of length $nt$ over $\mathbb{F}_p$, where each word is divided into $n$ blocks of length $t$.
- Important: the Hamming metric is still $q$-ary: when counting the weight or the distance, we consider an element of $\mathbb{F}_p^t$ as one symbol. Example ($p = 2$, $q = 2^3 = 8$): 000 <u>001</u> <u>110</u> 000 <u>111</u>

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# check matrix

- Every subspace of $\mathbb{F}_p^{nt}$ can be represented by a generator matrix, whose rows form basis.
- Every subspace $C$ of the space $nt$-words over $\mathbb{F}_p$ can be represented as the null-space, $\mathrm{null}(M)$, of an $(nt - \dim C) \times nt$ matrix $M$, called a check matrix of the $\mathbb{F}_p$-linear code $C$.
- Since we consider the positions of $nt$-words as arranged into blocks, the columns of a check matrix are also naturally grouped into $n$ groups.

$$H = \begin{pmatrix} 00 & 00 & 01 & 01 & 10 & 10 & 01 & 01 & 10 & 01 & 01 & 10 & 00 \\ 01 & 01 & 10 & 10 & 00 & 00 & 01 & 10 & 01 & 11 & 11 & 10 & 00 \\ 10 & 10 & 00 & 00 & 01 & 01 & 10 & 01 & 01 & 10 & 10 & 10 & 00 \end{pmatrix}$$

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# Connection of one-weight codes and intriguing sets

**Lemma**

*Assume that an* $m \times nt$ *matrix* $M$ *over* $\mathbb{F}_p$ *is a generator matrix if the code* $C$ *and a check matrix of the code* $C^{\perp}$. $C$ *is a one-weight code with weight* $w$ *if and only if* $C^{\perp}$ *is an intriguing set with quotient matrix* $\begin{pmatrix} \cdot & \cdot \\ \mu & \cdot \end{pmatrix}$, *where* $w \cdot p^{t-1} = \mu \cdot p^{m-1}$.

# Treating a group $T$ of columns in a check matrix

- For a given finite multiset $T$ of vectors, by $\langle\langle T \rangle\rangle$ we denote the multiset
$$\langle\langle T \rangle\rangle := \left\{ \sum_{v \in T} a_v v : \ a_v \in \mathbb{F}_p \right\}$$
of all $q^{|T|}$ linear combinations of elements from $T$.
- Every such $\langle\langle T \rangle\rangle$ will be called a multisubspace, or $t$-multisubspace, (of the vector space) with a "basis" $T$.
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# Multispreads

- We will call a collection $(S_1, \ldots, S_n)$ of $t$-multisubspaces of $\mathbb{F}_p^m$ a $(\lambda, \mu)$-multispread, or multispread, if there hold

$$S_1 \uplus \ldots \uplus S_n = (n+\lambda)\times\{\bar{0}\} \uplus \mu\times\mathbb{F}_p^{m*}.$$

- or, equivalently,

$$S_1^* \uplus \ldots \uplus S_n^* = \lambda\times\{\bar{0}\} \uplus \mu\times\mathbb{F}_p^{m*},$$

where ${S_i}^* = S_i - \{\bar{0}\}$.

- $(0,\mu)$-multispreads are known as $\mu$-fold spreads (in this case, $S_i$ are ordinary $t$-dimensional subspaces, without multiplicity larger than 1);
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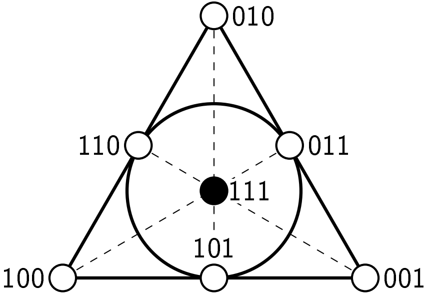
# Example: $(1,2)$-multispread, $\mathbb{F}_2^3$, $t=2$.

An example of a $(1,2)$-multispread in $\mathbb{F}_2^3$ from four 2-subspaces $\langle\langle 100, 010\rangle\rangle$, $\langle\langle 100, 001\rangle\rangle$, $\langle\langle 010, 001\rangle\rangle$, $\langle\langle 110, 011\rangle\rangle$, and one 1-subspace $\langle\langle 111, 000\rangle\rangle$ (having multiplicity 2, as a multisubspace).



# Multispreads $\leftrightarrow$ additive intriguing sets (CR-1)

**Theorem**

*Assume $M$ is an $m \times nt$ matrix over $\mathbb{F}_p$ with groups of columns $T_1, \ldots, T_n$ (each group has $t$ columns). The code $\mathrm{null}(M)$ is an $\mathbb{F}_p$-linear intriguing set in the Hamming space $H(n,q)$, $q = p^t$, with quotient matrix*

$$\begin{pmatrix} \lambda & n(q-1)-\lambda \\ \mu & n(q-1)-\mu \end{pmatrix}$$

*if and only if $\{\langle\langle T_1 \rangle\rangle, \ldots, \langle\langle T_n \rangle\rangle\}$ is a $(\lambda,\mu)$-multispread, i.e.,*

$$\langle\langle T_1 \rangle\rangle^* \uplus \ldots \uplus \langle\langle T_n \rangle\rangle^* = \lambda \times \{\bar{0}\} \uplus \mu \times \mathbb{F}_p^{m*}.$$

- The famous Bonisoli theorem [1] that characterizes linear one-weight codes corresponds to $t = 1$, i.e., $q = p$.

[1] Bonisoli, A. Every Equidistant Linear Code Is a Sequence of Dual Hamming Codes, Ars Comb. 1984

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# A necessary condition and Wrong conjecture

$$S_1^* \uplus \ldots \uplus S_n^* = \lambda \times \{\bar{0}\} \uplus \mu \times \mathbb{F}_p^{m*},$$

- CONJECTURE. Assume that $t \le m$ and $p \le \mu$. A $(\lambda, \mu)$-multispread exists if and only if

$$\lambda + \mu(p^m - 1) \text{ is divisible by } p^t - 1;$$

  that is, if and only if

$$\lambda \equiv -\mu(p^m - 1) \bmod p^t - 1.$$

- This divisibility condition is necessary but (oops!) not sufficient.
- Denote by $\lambda_{\min} = \lambda_{\min}(p, t, m, \mu)$ the smallest nonnegative integer $\lambda$ satisfying the divisibility condition.

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# The divisibility condition is NOT sufficient

## Lemma (additional necessary condition for small $\mu$)

*If a $(\lambda, \mu)$-multispread exists, then there is an integer $n_0$ such that*

$$\mu \frac{q^m - 1}{q^t - 1} \leq n_0 \leq \mu \frac{q^m - 1}{q^t - q^{i_{\mathrm{mx}}}}$$

*where* $i_{\mathrm{mx}} = \min\{\lfloor \log_q(\mu) \rfloor, t - 1\}$.

## Corollary

*For any $\lambda$, there are no $(\lambda, 2)$- and $(\lambda, 3)$-multispreads from 4-multisubspaces of $\mathbb{F}_2^5$.*

# The divisibility condition is NOT sufficient: EXAMPLE

**Corollary**

*For any $\lambda$, there are no $(\lambda, 2)$- and $(\lambda, 3)$-multispreads from 4-multisubspaces of $\mathbb{F}_2^5$.*

- Proof for $\mu = 3$.

$$\lambda \overset{\mathrm{mod}\ p^t-1}{\equiv} -\mu(p^m - 1) = -3 \cdot (2^5 - 1) = -93 \overset{\mathrm{mod}\ 15}{\equiv} 12.$$

Since $\mu = 3 < 2^2$, every 4-multisubspace from the multispread has

- rank 4 (contributes 0 to $\lambda$), or
- rank 3 (contributes 1 to $\lambda$), or
- rank 0 (contributes 15 to $\lambda$).

We see that the multispread has at least 12 multisubspaces of rank 3. They cover at least $12 \cdot 14 = 168$ nonzero points, which is larger than $\mu \times |\mathbb{F}_2^{5*}| = 3 \times 31 = 93$, a contradiction.

# Special case: $t \geq m$

**Theorem**

*A $(\lambda, \mu)$-multispread, $t \geq m$ exists if and only if $\mu$ is divisible by $p^{t-m}$ and*

$$\lambda = \lambda_{\min} + \ell(p^t - 1) = (p^{t-m} - 1)\frac{\mu}{p^{t-m}} + \ell(p^t - 1)$$

*for some nonnegative integer $\ell$.*

- *Sufficiency.* Take $\mu/p^{t-m}$ times the $t$-multisubspace that spans the whole $m$-dimensional space and $\ell$ times the trivial $t$-multisubspace of rank-0.
- *Nessecity.* Since the maximum rank of a multisubspace is $m$, all multiplicities are divisible by $p^{t-m}$. The remaining is the divisibility condition

# Special case: $\mu < p$

**Theorem**

*A $(\lambda, \mu)$-multispread such that $\mu < p$ exists if and only if $\mu$ is divisible by $\frac{p^t-1}{p^s-1}$, where $s = \gcd(t, m)$, and $\lambda$ is divisible by $p^t - 1$.*

- If $\mu < p$, then a $(\lambda, \mu)$-multispread can only consist of multisubspaces of rank 0 and $t$. Hence, the collection of $t$-subspaces forms a $\mu$-fold spread, while the multisubspaces of rank 0 forms a $(\lambda = \ell(p^t - 1), 0)$-multispread (where $\ell$ is the number of 0-subspaces).

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# Basic constructions of multispreads

**Lemma**

*If $A$ and $B$ are $(\lambda, \mu)$- and $(\lambda', \mu')$-multispreads from $t$-multisubspaces of $\mathbb{F}_p^m$, then $A \uplus B$ is a $(\lambda + \lambda', \mu + \mu')$-multispread.*

# Basic constructions of multispreads

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $M$ $t$-multisubspaces of $\mathbb{F}_p^m$, $M = \frac{\mu(p^m-1)+\lambda}{p^t-1}$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(a) $m' = m, \quad t' = t, \quad \lambda' = \lambda + p^t - 1, \quad \mu' = \mu$;

(b) $m' = m, \quad t' = t, \quad \lambda' = \lambda, \quad \mu' = \mu + \frac{p^t-1}{p^s-1}$, where $s = \gcd(t, m)$;

(c) $m' = m + t, \quad t' = t, \quad \lambda' = \lambda, \quad \mu' = \mu$;

(d) $m' = m - 1, \quad t' = t, \quad \lambda' = \lambda + (p-1)\mu, \quad \mu' = p\mu$;

(e) $m' = m, \quad t' = t + 1, \quad \lambda' = p\lambda + (p-1)M, \quad \mu' = p\mu$;

(f) $p' = p^{\frac{1}{s}}, \quad m' = ms, \quad t' = ts, \quad \lambda' = \lambda, \quad \mu' = \mu$, where $p = p'^s$.

# Basic constructions: increasing $\lambda$

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $t$-multisubspaces of $\mathbb{F}_p^m$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(a) $m' = m, \quad t' = t, \quad \underline{\lambda' = \lambda + (p^t - 1)}, \quad \mu' = \mu;$

Proof: It follows from the existence of a $(p^t - 1, 0)$-multispread, which consists of one $t$-multisubset of rank 0.

# Basic constructions: increasing $\mu$

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $t$-multisubspaces of $\mathbb{F}_p^m$, $t \leq m$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(b) $m' = m$, $t' = t$, $\lambda' = \lambda$, $\underline{\mu' = \mu + \frac{p^t-1}{p^s-1}}$, *where* $s = \gcd(t, m)$;

Proof: It follows from the existence of a $(0, \frac{p^t-1}{p^s-1})$-multispread (known as $\frac{p^t-1}{p^s-1}$-fold spread in the literature).

# Basic constructions: increasing $m$

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $t$-multisubspaces of $\mathbb{F}_p^m$, $t \leq m$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(c) $\underline{m' = m + t}$, $\quad t' = t, \quad \lambda' = \lambda, \quad \mu' = \mu;$

*Proof*: Use a known partition of $\mathbb{F}_p^{m'} = \mathbb{F}_p^{m+t}$ into an $m$-subspace and $t$-subspaces [2]:

- Let $U$ be a $t$-subspace (over $\mathbb{F}_p$) of $\mathbb{F}_{p^m}$.
  Partition of $\mathbb{F}_{p^m} \times U \sim \mathbb{F}_p^{m+t}$:
  - $\{(\alpha u|u) : u \in U\}$ — $t$-subspace for each $\alpha \in \mathbb{F}_{p^m}$;
  - $\{(v|\mathbf{0}) : v \in \mathbb{F}_{p^m}\}$ — $m$-subspace.

[2] T. Bu. Partitions of a vector space. Discrete Math., 31:79–83, 1980.

# Basic constructions: projection

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $t$-multisubspaces of $\mathbb{F}_p^m$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(d) $\underline{m' = m - 1}$, $t' = t$, $\underline{\lambda' = \lambda + (p-1)\mu}$, $\underline{\mu' = p\mu}$.

*Proof*: Projection: $\mathbb{F}_p^m \to \mathbb{F}_p^{m-1}$ by removing the last component.

- $t$-multisubspace of $\mathbb{F}_q^m \longrightarrow t$-multisubspace of $\mathbb{F}_p^{m-1}$.
- Since the preimage of every non-zero vector in $\mathbb{F}_p^{m-1}$ is $p$ non-zero vectors in $\mathbb{F}_p^m$, we get $\underline{\mu' = p\mu}$.
- Since the preimage of the zero vector in $\mathbb{F}_p^{m-1}$ is the zero vector and $p-1$ non-zero vectors in $\mathbb{F}_p^m$, we get $\underline{\lambda' = \lambda + (p-1)\mu}$.

# Basic constructions: increasing $t$

**Lemma**

*If there is a $(\lambda, \mu)$-multispread from $M$ $t$-multisubspaces of $\mathbb{F}_p^m$, $M = \frac{\mu(p^m-1)+\lambda}{p^t-1}$, then there is a $(\lambda', \mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_p^{m'}$, where*

(e) $m' = m, \quad t' = t + 1, \quad \lambda' = p\lambda + (p-1)M, \quad \mu' = p\mu;$

*Proof*: (e) is obtained if we treat $t$-multisubspaces as $(t+1)$-multisubspaces, with the corresponding multiplicities of vectors.

# Basic constructions: treatment for subfield

**Lemma**

*If there is a $(\lambda,\mu)$-multispread from t-multisubspaces of $\mathbb{F}_p^m$ and $p = p'^s$, then there is a $(\lambda',\mu')$-multispread from $t'$-multisubspaces of $\mathbb{F}_{p'}^{m'}$, where*

(f) $p' = p^{1/s}, \quad m' = ms, \quad t' = ts, \quad \lambda' = \lambda, \quad \mu' = \mu.$

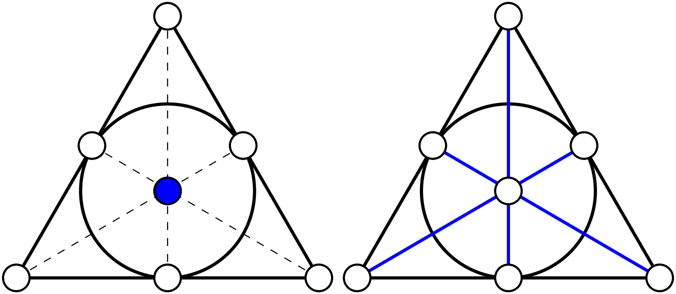
Proof: treat $\mathbb{F}_p^k$ as $\mathbb{F}_{p'}^{sk}$.

# Switching construction

By removing an $(m-2)$-dimensional subspace $S$ and adding all $p+1$ $(m-1)$-dimensional subspaces including $S$, we keep the property of being a multispread, but change the parameters:

$$(\lambda, \mu) \quad \to \quad (\lambda-(p-1),\, \mu+1).$$

The inverse operation is also useful.



# Special construction: from the Desarguesian spread in $\mathbb{F}_{p^6}$

- $\mathbb{F}_{p^3} \subset \mathbb{F}_{p^6}$; $\bar{D} = \{\mathbb{F}_{p^3}, a_1\mathbb{F}_{p^3}, \ldots, a_{p^3}\mathbb{F}_{p^3}\}$ — a partition of $\mathbb{F}_{p^6}$.
- Let $T$ be a 4-dimensional $\mathbb{F}_p$-subspace of $\mathbb{F}_{p^6}$ that intersects with $p+1$ subspaces from $\bar{D}$ (call them $\bar{D}^2_T$) in a 2-dimensional subspace and the other $p^3-p$ subspaces from $\bar{D}$ (call them $\bar{D}^1_T$) in a 1-dimensional subspace.
- $\bar{D}$ itself is a $(\lambda_0, \mu = p)$-multispread, $\lambda_0 = (p-1)(p^3+1)$;
- $\mathrm{Orbit}_{\mathbb{F}^\times_{p^3}}(T) \cup (\bar{D} - \bar{D}^2_T)$ is a $(\lambda_1, \mu = p+1)$-multispread;
- $\mathrm{Orbit}_{\mathbb{F}^\times_{p^3}}(T_1) \cup \mathrm{Orbit}_{\mathbb{F}^\times_{p^3}}(T_2) \cup (\bar{D} - \bar{D}^2_{T_1} - \bar{D}^2_{T_2})$ is a $(\lambda_2, \mu = p+2)$-multispread if $\bar{D}^2_{T_1}$ and $\bar{D}^2_{T_2}$ are disjoint;
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# $t = 2$: $p^2$-ary additive one-weight codes

**Theorem**

*Conjecture holds for $t = 2$ and any $m \geq 2$ (any $p$).*

- *Sketch:* It is sufficient to construct $(\lambda_{\min}, \mu)$-multispread for $m = 3$ and $\mu \in \{p, p+1, \ldots, 2p\}$.
  - $(\lambda = p-1, \mu = p)$: the projection of a spread of $\mathbb{F}_p^4$.
  - $(\lambda = 0, \mu = p+1)$: multifold spread consisting of all 2-subspaces.
  - $(\lambda = 0, \mu = 2p+2)$: multifold spread consisting of all 2-subspaces with multiplicity 2.
  - $(\lambda_{\min}, \mu = 2p+2-i)$, $i = 1, 2, ..., p$: switching by replacing a pencil of projective lines through a projective point $x$ by $x$ (we need $i$ such points, no 3 of them lying on one line).

# $t = 3$: 8-, 27-ary additive one-weight codes

**Theorem**

*Conjecture holds for $t = 3$ and any $m \equiv 1 \bmod t$ (any $p$).*

- *Sketch:* Consider a spread of $\mathbb{F}_p^4$ into $p^2+1$ subspaces of dimension 2. If we treat 2-subspaces as 3-multisubspaces (basic construction (e)), we get a $(\lambda_{\min} = (p^2+1)(p-1), \mu = p)$-multispread. Then, by switching (replacing a multisubspace of rank 2 by $p+1$ multisubspace of rank 3) we increase $\mu$ one-by-step.

**Theorem (involves computations for $p=2$, $\mu=3$ and $p=3$, $\mu=4,5$)**

*Conjecture holds for $t = 3$ and any $m \equiv 2 \bmod t$, $p = 2, 3$.*

- Note that for $m \equiv 0 \bmod t$, all parameters are solved by (ordinary) $t$-spreads.

# $t = 3$: 8-, 27-ary additive one-weight codes

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*Conjecture holds for $t = 3$ and any $m \equiv 2 \bmod t$, $p = 2, 3$.*

- Note that for $m \equiv 0 \bmod t$, all parameters are solved by (ordinary) $t$-spreads.

# $t = 4$: 16-ary additive one-weight codes

**Theorem (special construction)**

*Conjecture holds for $t = 4$ and any $m \equiv 2 \bmod 4$ (any $p$).*

**Theorem (involves computations for $(\lambda, \mu) = (9, 3)$)**

*Conjecture holds for $t = 4$ and any $m \equiv 3 \bmod 4$, $p = 2$.*

**Theorem (involves computations for $\mu \in \{2, 3\}$, $m = 9$)**

*Conjecture holds for $t = 4$ and any $m \equiv 1 \bmod 4$, $p = 2$, except for $m = 5$, $\mu \in \{2, 3\}$.*

- In the last case, for $\mu \in \{2, 3\}$, $m = 5$ conjecture fails, and we needed to solve $m = 5 + t = 9$ (solved with computer and prescribed Aut).

# First problem for $t=3$, $m=5$ and in general for $m=2t-1$

1st open question: $\mu = p+1$, $\lambda = (p^2+1)(p-1)$

| $p$ | $\lambda$ | $\mu$ | 2-dim | 3-dim | comment |
|---|---|---|---|---|---|
| $p$ | $(p^2+1)(p-1)$ | $p+1$ | $p^2+1$ | $p^3+1$ | |
| 2 | 5 | 3 | 5 | 9 | ∃, ILP [3] |
| 3 | 20 | 4 | 10 | 28 | ∃, ILP [4] |
| 4 | 51 | 5 | 17 | 65 | ? |
| 5 | 104 | 6 | 26 | 126 | ? |

- *Problem*: find $p^{t-1}+1$ $(t-1)$-subspaces and $p^t+1$ $t$-subspaces of $\mathbb{F}_p^{2t-1}$ ($p$ prime power) such that each nonzero point belongs to $p+1$ chosen $t$-subspaces or to 1 chosen $t$-subspace and 1 chosen $(t-1)$-subspace.
- *Equivalent dual* (see the next slide) *problem* : find $p^{t-1}+1$ $t$-subspaces and $p^t+1$ $(t-1)$-subspaces of $\mathbb{F}_p^{2t-1}$ such that each nonzero point belongs to exactly 2 chosen subspaces.

[3] $\max|\mathrm{Aut}| = 6$

[4] $\max|\mathrm{Aut}| = 6$ (Sascha Kurz, private communication)

# Duality

**Theorem (special case of [a])**

[a] [S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, C. Vanden Eynden. On $\lambda$-fold partitions of finite vector spaces and duality. Discrete Math., 2011]

*A multiset $S = \{C_1, \ldots, C_n\}$ from t-multisubspaces of $\mathbb{F}_p^m$ is a $(\lambda, \mu)$-multispread if and only if $\{C_1^\perp, \ldots, C_n^\perp\}$ is a $\nu$-fold partition of $\mathbb{F}_p^m$, where*

$$\nu = n - p^{m-t}\mu$$

*or, equivalently,*

$$(p^t - 1)\nu = (p^{m-t} - 1)\mu + \lambda.$$

# Problems

- Construct multispreads with new parameters (not necessarily the first unsolved)
- Characterize infinite sequences of parameters.
- (difficult) Construct multispreads with given ranks of subspaces, characterize admissible parameters (including the collection of ranks).

# NEW: Connection with mixed orthogonal arrays

(joint work with Ferruh Özbudak and Vladimir Potapov)

- OA$(M, q_1 \cdot q_2 \cdot \ldots \cdot q_n, t)$ is code $C$ in $[q_1] \times [q_2] \times \ldots \times [q_m]$ such that fixing the values in any $t$ different positions $i_1$, $i_2$, ..., $i_t$ we always get $\frac{|C|}{q_{i_1} q_{i_2} \cdots q_{i_t}}$ codewords.
- Equivalently, it is an algebraic $t$-design (avoiding $t$ largest nonmain eigenvalues) in the multigraph

$$H = \frac{Q}{q_1} K_{q_1} \square \frac{Q}{q_2} K_{q_2} \square \cdots \square \frac{Q}{q_n} K_{q_n}, \quad \text{where } Q = \mathrm{l.c.m.}(q_1, ..., q_n).$$

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# Mixed OA attaining the generalized B.-F. bound

- The generalized Bierbrouer–Friedman bound for algebraic $t$-designs in arbitrary regular multigraph and for multigraph $H$ above:

$$\frac{|C|}{|G|} \geq \frac{-\theta_{t+1}}{\theta_0 - \theta_{t+1}}$$

$$\frac{|C|}{q_1 q_2 \dots q_n} \geq 1 - (1 - \frac{1}{q})\frac{n}{t+1}$$

where $q$ is the harmonic mean of all $q_i$, $i = 1, ..., n$.

- The algebraic designs satisfying the g. B.-F. bound above are $\equiv$ independent intriguing sets.
- Additive independent intriguing sets in $H$, where $q_i = p^{r_i}$ correspond to $t$-multispreads in $\mathbb{F}_p^m$, where $t = \max_i r_i$ ($r_i$ are the ranks of the corresponding subspaces).

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