Multispreads and additive intriguing sets in Hamming graphs

Denis Krotov
Sobolev Institute of Mathematics
dk@ieee.org

A set C of vertices of a connected regular graph is called *intriguing* if both C and its complement $\overline{C}$ induce regular subgraphs. Nontrivial (when both C and $\overline{C}$ are nonempty) intriguing sets are special cases of completely regular codes (namely, intriguing sets are completely regular codes with covering radius 1), introduced by Delsarte [1] for distance-regular graphs and redefined by Neumaier [2] in a manner applicable to an arbitrary graph. We show that in a Hamming graph over a field alphabet, additive (closed under addition) intriguing sets are equivalent to some multiset generalization of spreads, defined below.

Let S be a collection (multiset) of subspaces of dimension at most t of an m-dimensional space over $\mathrm{GF}(p)$. Each subspace X of dimension s from S is treated as the multiset of cardinality p^t-1 where every nonzero vector of X has multiplicity p^{t-s} and the zero vector has multiplicity $p^{t-s}-1$. Such S is called a (λ,μ)-*multispread* (more specifically, a $(\lambda,\mu)_p^{t,m}$-multispread) if the union of the multisets corresponding to the subspaces from S contains the zero vector with multiplicity λ and each nonzero vector of the space with multiplicity μ.

Ordinary spreads correspond to $(0,1)$-multispreads, and μ-fold spreads correspond to $(0,\mu)$-multispreads [3, p. 83]. An example of a (λ,μ)-multispread with nonzero $\lambda=p^{m'-m}-1$ and $\mu=p^{m'-m}$ can be obtained from a spread of an m'-dimensional space, $m'>m$, by projection onto an m-dimensional space (we consider the projection that respects the multiplicity and preserves the cardinality of a multiset of vectors).

Multispreads can be considered as a special case of vector-space partitions [4], and the subspaces dual to the subspaces from a multispread also form a multifold partition of the space, dual to the original multispread.

The current work is devoted to the characterization of the parameters of multispreads, which is equivalent (for prime p) to the characterization of the parameters of additive intriguing sets in the Hamming graphs over $\mathrm{GF}(p^t)$ and also (via duality) to the characterization of the parameters of additive one-weight codes over $\mathrm{GF}(p^t)$. We characterize these parameters for the case $t=2$ and make a partial characterization for $t=3$ and $t=4$ (including a complete characterization for $p^t=2^3$, 3^3, and 2^4, where several key cases are solved computationally).

This is joint work with Ivan Mogilnykh. The work is funded by the Russian Science Foundation (22-11-00266).

References

[1] P. Delsarte. An Algebraic Approach to Association Schemes of Coding Theory, volume 10 of Philips Res. Rep., Supplement. Eindhoven, 1973.

[2] A. Neumaier, Completely regular codes. *Discrete Math.* **106–107** (1992) 353–360. https://doi.org/10.1016/0012-365X(92)90565-W

[3] J. W. P. Hirschfeld. Projective Geometries Over Finite Fields. Oxford University Press, New York, 1979.

[4] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden, On λ-fold partitions of finite vector spaces and duality. *Discrete Math.* **311**(4) (2011) 307–318. https://doi.org/10.1016/j.disc.2010.10.026