On recent progress in distance-regular graphs

Jack Koolen*

*School of Mathematical Sciences
University of Science and Technology of China

(Based on joint work with Hongjun Ge, Chenhui Lv, Qianqian Yang, Alexander Gavrilyuk, Mamoon Abdullah, Brhane Gebremichel, Jae-Ho Lee, Shuandong Li, Yun-Han Li, Xiaoye Liang and Ying Ying Tan. Some of it is still ongoing.)

G2C2
August 11–25, 2024

Outline

1. Distance-regular graphs
   - Definitions
   - Examples
   - DRG with classical parameters
2. DRG with classical parameters
   - Known classification results
3. Bounding the parameters α, β
   - α
   - β
4. 1-Homogeneous graphs
   - 1-Homogeneous graphs
   - Main result
5. Thin DRG
   - Definitions

Outline

Notations

Definition

Graph: $\Gamma = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

- All graphs in the talk are undirected and simple (no loops or multiple edges).

Notations

Definition

Graph: $\Gamma = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

- All graphs in the talk are undirected and simple (no loops or multiple edges).
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by its vertices, such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.
- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.

Notations

Definition

Graph: $\Gamma = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

- All graphs in the talk are undirected and simple (no loops or multiple edges).
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by its vertices, such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.
- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.
- $d(x, y)$:the distance between x and y.
- $D(\Gamma)$: diameter of Γ, if Γ is connected.

Notations

Definition

Graph: $\Gamma = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

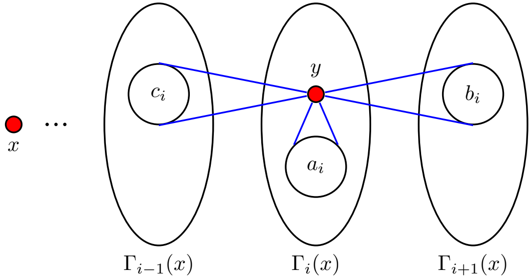
- All graphs in the talk are undirected and simple (no loops or multiple edges).
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by its vertices, such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.
- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.
- $d(x, y)$:the distance between x and y.
- $D(\Gamma)$: diameter of Γ, if Γ is connected.
- $\Gamma_i(x)$= $\{y \mid d(x, y) = i\}$, $\Gamma(x) = \Gamma_1(x) = \{y \mid x \sim y\}$.
- The subgraph induced on $\Gamma(x)$ is the local graph of Γ at x, denoted by $\Delta(x)$.

Distance-regular graphs (DRG)

A connected graph Γ is called distance-regular (DRG) if there are numbers a_i, b_i, c_i, $0 \leq i \leq D(\Gamma)$, such that for any two vertices x and y with $d(x, y) = i$,

$$|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = c_i, \ |\Gamma_1(y) \cap \Gamma_i(x)| = a_i, \ |\Gamma_1(y) \cap \Gamma_{i+1}(x)| = b_i.$$

a_i, b_i, c_i, $0 \leq i \leq D(\Gamma)$ are called the intersection numbers of Γ.



Distance-regular graphs (DRG)

Γ is regular with valency $b_0 = |\Gamma_1(x)|$ for any $x \in V(\Gamma)$, and

$$b_0 = c_i + a_i + b_i.$$

Distance-regular graphs (DRG)

Γ is regular with valency $b_0 = |\Gamma_1(x)|$ for any $x \in V(\Gamma)$, and

$$b_0 = c_i + a_i + b_i.$$

For a DRG Γ with diameter D, its intersection array is

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

Outline

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- The Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- The Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.
- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- The Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.
- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.
- Gives an algebraic frame work to study codes, especially bounds on codes.
- For example: the Delsarte linear programming bound and the Schrijver bound.

Johnson graphs

Definition

- $1 \leq t \leq n$ integers.
- $N = \{1, \ldots, n\}$
- The Johnson graph $J(n, t)$ has vertex set $\binom{N}{t}$
- $A \sim B$ if $\# A \cap B = t - 1$.

Johnson graphs

Definition

- $1 \leq t \leq n$ integers.
- $N = \{1, \ldots, n\}$
- The Johnson graph $J(n, t)$ has vertex set $\binom{N}{t}$
- $A \sim B$ if $\#A \cap B = t - 1$.
- $J(n, t) \cong J(n, n - t)$, diameter $\min\{t, n - t\}$.
- DRG with $c_i = i^2$.

Johnson graphs

Definition

- $1 \leq t \leq n$ integers.
- $N = \{1, \ldots, n\}$
- The Johnson graph $J(n, t)$ has vertex set $\binom{N}{t}$
- $A \sim B$ if $\# A \cap B = t - 1$.
- $J(n, t) \cong J(n, n - t)$, diameter $\min\{t, n - t\}$.
- DRG with $c_i = i^2$.
- Gives an algebraic frame work to study designs.

More examples

- Many known infinite families (with unbounded diameter) of distance-regular graphs come from classical objects, for example:

More examples

- Many known infinite families (with unbounded diameter) of distance-regular graphs come from classical objects, for example:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - dual polar graphs (The vertices are the maximal totally isotropic subspaces on a vector space over a finite field with a fixed (non-degenerate) bilinear form).

More examples

- Many known infinite families (with unbounded diameter) of distance-regular graphs come from classical objects, for example:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - dual polar graphs (The vertices are the maximal totally isotropic subspaces on a vector space over a finite field with a fixed (non-degenerate) bilinear form).
- Distance-regular graphs give a way to study these classical objects from a unified combinatorial view point.

Some more examples

There are four known infinite families (with unbounded diameter) of vertex-transitive but not distance-transitive DRGs, namely

- the Ustimenko graphs;
- the Hemmeter graphs;
- the Doob graphs;
- the quadratic forms graphs.

Some more examples

There are four known infinite families (with unbounded diameter) of vertex-transitive but not distance-transitive DRGs, namely

- the Ustimenko graphs;
- the Hemmeter graphs;
- the Doob graphs;
- the quadratic forms graphs.

There is only one infinite family known (with unbounded diameter) which is not vertex-transitive, namely the twisted Grassmann graphs (discovered in 2005 by Van Dam and K.). They have two orbits under their full automorphism group.

Outline

We say that a distance-regular graph Γ of diameter D has *classical parameters* (D, b, α, β) if the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), \tag{1}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \tag{2}$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geq 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b = 0$.

We say that a distance-regular graph Γ of diameter D has *classical parameters* (D, b, α, β) if the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), \qquad (1)$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \qquad (2)$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geq 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b = 0$.
We note that $b \neq 0, -1$ by the following result.

Lemma

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and the diameter $D \geq 3$. Then, b is an integer such that $b \neq 0, -1$.

There are many examples of DRG with classical parameters namely:

- Hamming graphs and Doob graphs,
- Johnson graphs,
- Grassmann graphs and twisted Grassmann graphs,
- bilinear forms graphs,
- sesquilinear forms graphs,
- quadratic forms graphs,
- dual polar graphs,
- the Ustimenko graphs,
- the Hemmeter graphs.

- Bannai's problem is to classify *Q*-polynomial distance-regular graph with large diameter.

- Bannai's problem is to classify *Q*-polynomial distance-regular graph with large diameter.
- For diameter 3 there are too many examples.

- Bannai's problem is to classify *Q*-polynomial distance-regular graph with large diameter.
- For diameter 3 there are too many examples.
- An important subproblem of Bannai's problem is to classify the DRG with classical parameters, as they are *Q*-polynomial.

- Bannai's problem is to classify Q-polynomial distance-regular graph with large diameter.
- For diameter 3 there are too many examples.
- An important subproblem of Bannai's problem is to classify the DRG with classical parameters, as they are Q-polynomial.
- This is a very hard problem as the twisted Grassmann graphs do exist.

- Bannai's problem is to classify Q-polynomial distance-regular graph with large diameter.
- For diameter 3 there are too many examples.
- An important subproblem of Bannai's problem is to classify the DRG with classical parameters, as they are Q-polynomial.
- This is a very hard problem as the twisted Grassmann graphs do exist.
- All the known infinite families of DRG with valency at least three and with unbounded diameter have classical parameters or are very closely related to an infinite family of DRG with classical parameters, like the folded hypercubes and the doubled Grassmann graphs.

- If b is negative then they are essentially classified by C.-W. Weng. There is still one infinite family of feasible parameter sets, for which we do not have any idea whether they exist or not.

- If b is negative then they are essentially classified by C.-W. Weng. There is still one infinite family of feasible parameter sets, for which we do not have any idea whether they exist or not.
- Terwilliger in the 1980's classified the DRG with classical parameters with $b = 1$.

- If b is negative then they are essentially classified by C.-W. Weng. There is still one infinite family of feasible parameter sets, for which we do not have any idea whether they exist or not.
- Terwilliger in the 1980's classified the DRG with classical parameters with $b = 1$.
- He obtained:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) where $b = 1$ and $D \geq 4$. Then Γ is a Hamming graph, a halved cube, a Johnson graph or a Doob graph.

Outline

The Grassmann graphs

Theorem (Metsch(1995))

The Grassmann graphs $J_q(n, D)$ ($n \geq 2D$) are characterized by their intersection array if $n \geq \max\{2D+2, 2D+6-q\}$ and $D \geq 3$.

The Grassmann graphs

Theorem (Metsch(1995))

The Grassmann graphs $J_q(n, D)$ $(n \geq 2D)$ are characterized by their intersection array if $n \geq \max\{2D+2, 2D+6-q\}$ and $D \geq 3$.

- What happens for $n = 2D, n = 2D + 1$?

The Grassmann graphs

Theorem (Metsch(1995))

The Grassmann graphs $J_q(n, D)$ $(n \geq 2D)$ are characterized by their intersection array if $n \geq \max\{2D+2, 2D+6-q\}$ and $D \geq 3$.

- What happens for $n = 2D$, $n = 2D + 1$?
- Van Dam and K. (2005) found the twisted Grassmann graphs. They have the same parameters as $J_q(2D + 1, D)$, so the Grassmann graph $J_q(2D+1, D)$ is not determined by its intersection array.

- Gavrilyuk and K. (2024+): The Grassman graphs $J_q(2D, D)$ are uniquely determined if $D > 12$ or if $q \geq 9$ and $D > 7$.

- Gavrilyuk and K. (2024+): The Grassman graphs $J_q(2D, D)$ are uniquely determined if $D > 12$ or if $q \geq 9$ and $D > 7$.
- K., Lv and Gavrilyuk (in progress): The Grassmann graphs $J_2(2D + 3, D)$ are uniquely determined if $D \geq 3$.

- Gavrilyuk and K. (2024+): The Grassman graphs $J_q(2D, D)$ are uniquely determined if $D > 12$ or if $q \geq 9$ and $D > 7$.
- K., Lv and Gavrilyuk (in progress): The Grassmann graphs $J_2(2D + 3, D)$ are uniquely determined if $D \geq 3$.
- Mr. Lv talked about it last week.
- The method is a slight improvement of the method Metsch used.

- Gavrilyuk and K. (2024+): The Grassman graphs $J_q(2D, D)$ are uniquely determined if $D > 12$ or if $q \geq 9$ and $D > 7$.
- K., Lv and Gavrilyuk (in progress): The Grassmann graphs $J_2(2D+3, D)$ are uniquely determined if $D \geq 3$.
- Mr. Lv talked about it last week.
- The method is a slight improvement of the method Metsch used.
- Gavrilyuk (in progress): The Grassmann graphs $J_2(2D+2, D)$ are uniquely determined if $D \geq 3$ and D odd.

- Gavrilyuk and K. (2024+): The Grassman graphs $J_q(2D, D)$ are uniquely determined if $D > 12$ or if $q \geq 9$ and $D > 7$.
- K., Lv and Gavrilyuk (in progress): The Grassmann graphs $J_2(2D+3, D)$ are uniquely determined if $D \geq 3$.
- Mr. Lv talked about it last week.
- The method is a slight improvement of the method Metsch used.
- Gavrilyuk (in progress): The Grassmann graphs $J_2(2D+2, D)$ are uniquely determined if $D \geq 3$ and D odd.
- He uses the vanishing Krein parameters to obtain some extra conditions on the c_2-graph.

Bilinear forms graphs

Metsch, building on earlier work of Sprague, Ray-Chaudhuri, Huang and Cuypers, showed:

Theorem (Metsch (1999))

The bilinear forms graph Bil($D \times e, q$) is characterized by its intersection array if $q = 2$ and $e \geq D + 4$ or $q \geq 3$ and $e \geq D + 3$.

Bilinear forms graphs

Metsch, building on earlier work of Sprague, Ray-Chaudhuri, Huang and Cuypers, showed:

Theorem (Metsch (1999))

The bilinear forms graph Bil$(D \times e, q)$ is characterized by its intersection array if $q = 2$ and $e \geq D + 4$ or $q \geq 3$ and $e \geq D + 3$.

Gavrilyuk and K. (2018): The bilinear forms graph Bil$(D \times e, q)$ is characterized by its intersection array if $q = 2$ and $e = D$.

The following is a simplification of a result of Metsch.

The following is a simplification of a result of Metsch.

K. Metsch, 1999

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 2$ and it is not a Grassmann graph, or a bilinear forms graph. Then β is bounded by $b^{2D+4}(\alpha + 1)^2$.

The following is a simplification of a result of Metsch.

K. Metsch, 1999

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 2$ and it is not a Grassmann graph, or a bilinear forms graph. Then β is bounded by $b^{2D+4}(\alpha + 1)^2$.
Moreover, we have the following:
If $\alpha \in \{b - 1, b\}$, then $\beta \leq b^{D+3}$.

The following is a simplification of a result of Metsch.

K. Metsch, 1999

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 2$ and it is not a Grassmann graph, or a bilinear forms graph. Then β is bounded by $b^{2D+4}(\alpha + 1)^2$.
Moreover, we have the following:
If $\alpha \in \{b - 1, b\}$, then $\beta \leq b^{D+3}$.

Remark

Can we improve the bound for β?

The following is a simplification of a result of Metsch.

K. Metsch, 1999

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 2$ and it is not a Grassmann graph, or a bilinear forms graph. Then β is bounded by $b^{2D+4}(\alpha + 1)^2$.
Moreover, we have the following:
If $\alpha \in \{b - 1, b\}$, then $\beta \leq b^{D+3}$.

Remark

Can we improve the bound for β?
We think the upper bound for β should be something like Cb^D where C is a constant only depending on α and b and not on D.

The following is a simplification of a result of Metsch.

K. Metsch, 1999

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 2$ and it is not a Grassmann graph, or a bilinear forms graph. Then β is bounded by $b^{2D+4}(\alpha + 1)^2$.
Moreover, we have the following:
If $\alpha \in \{b - 1, b\}$, then $\beta \leq b^{D+3}$.

Remark

Can we improve the bound for β?
We think the upper bound for β should be something like Cb^D where C is a constant only depending on α and b and not on D.
Can we obtain a bound of α in terms of b?

Outline

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1+\alpha)(1+b)$ and hence $\alpha(b+1)$ is an integer.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1 + \alpha)(1 + b)$ and hence $\alpha(b + 1)$ is an integer.
- By looking at the integrability of c_3 and c_2 we easily obtain $\alpha \geq 0$.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1 + \alpha)(1 + b)$ and hence $\alpha(b + 1)$ is an integer.
- By looking at the integrability of c_3 and c_2 we easily obtain $\alpha \geq 0$.
- By looking at the integrability of p^4_{22} we obtain $\alpha \leq 5b^8$, if $D \geq 4$.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1+\alpha)(1+b)$ and hence $\alpha(b+1)$ is an integer.
- By looking at the integrability of c_3 and c_2 we easily obtain $\alpha \geq 0$.
- By looking at the integrability of p^4_{22} we obtain $\alpha \leq 5b^8$, if $D \geq 4$.
- All the known infinte unbounded diameter families of DRG with classical parameters have $\alpha \leq b + \sqrt{b}$.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1 + \alpha)(1 + b)$ and hence $\alpha(b + 1)$ is an integer.
- By looking at the integrability of c_3 and c_2 we easily obtain $\alpha \geq 0$.
- By looking at the integrability of p^4_{22} we obtain $\alpha \leq 5b^8$, if $D \geq 4$.
- All the known infinte unbounded diameter families of DRG with classical parameters have $\alpha \leq b + \sqrt{b}$.
- If $\alpha = 0$, $\beta \neq 0$, $b \geq 2$, $D \geq 4$ and locally the disjoint union of cliques, the graph must be a dual polar graph.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Then $c_2 = (1+\alpha)(1+b)$ and hence $\alpha(b+1)$ is an integer.
- By looking at the integrability of c_3 and c_2 we easily obtain $\alpha \geq 0$.
- By looking at the integrability of p^4_{22} we obtain $\alpha \leq 5b^8$, if $D \geq 4$.
- All the known infinte unbounded diameter families of DRG with classical parameters have $\alpha \leq b + \sqrt{b}$.
- If $\alpha = 0$, $\beta \neq 0$, $b \geq 2$, $D \geq 4$ and locally the disjoint union of cliques, the graph must be a dual polar graph.
- This result is based on work by Brouwer and Wilbrink, De Bruyn, Cameron, Cohen and others.

Our result on bounding the parameter α.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then $\alpha \leq b^2(b+1)$.

Our result on bounding the parameter α.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then $\alpha \leq b^2(b+1)$.
Moreover, if $b = 2$ and $D \geq 12$, then $\alpha \leq 2$.

Our result on bounding the parameter α.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then $\alpha \leq b^2(b+1)$.
Moreover, if $b = 2$ and $D \geq 12$, then $\alpha \leq 2$.

For $b = 2$, this shows that we have only 7 choices for α. Several of them can be removed by looking at the integrability of p^{2i}_{ii}. This is still work in progress with H. Ge, C. Lv, and Q. Yang.

Our result on bounding the parameter α.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then $\alpha \leq b^2(b+1)$.
Moreover, if $b = 2$ and $D \geq 12$, then $\alpha \leq 2$.

For $b = 2$, this shows that we have only 7 choices for α. Several of them can be removed by looking at the integrability of p^{2i}_{ii}. This is still work in progress with H. Ge, C. Lv, and Q. Yang. Mr. Ge talked about this earlier in this workshop.

Our result on bounding the parameter α.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then $\alpha \leq b^2(b+1)$.
Moreover, if $b = 2$ and $D \geq 12$, then $\alpha \leq 2$.

For $b = 2$, this shows that we have only 7 choices for α. Several of them can be removed by looking at the integrability of p^{2i}_{ii}. This is still work in progress with H. Ge, C. Lv, and Q. Yang. Mr. Ge talked about this earlier in this workshop.
As I already said, all the known DRG with classical parameters and positive b have $\alpha \leq b + \sqrt{b}$, if D is at least 8. We wonder whether this is the right bound.

Outline

Partial linear spaces

With C. Lv, we are working on bounding the parameter β in terms of D, b and α.

Partial linear spaces

With C. Lv, we are working on bounding the parameter β in terms of D, b and α.

Some definitions.

- An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively.

Partial linear spaces

With C. Lv, we are working on bounding the parameter β in terms of D, b and α.
Some definitions.

- An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively.
- If $(p, \ell) \in \mathcal{I}$ we say that p is *incident* with ℓ, or that p is on the line ℓ. The *order of a point* is the number of lines it is incident with and similarly for lines.

Partial linear spaces

With C. Lv, we are working on bounding the parameter β in terms of D, b and α.
Some definitions.

- An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively.
- If $(p, \ell) \in \mathcal{I}$ we say that p is *incident* with ℓ, or that p is on the line ℓ. The *order of a point* is the number of lines it is incident with and similarly for lines.
- The *point-line incidence matrix* of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the $|\mathcal{P}| \times |\mathcal{L}|$-matrix such that the (p, ℓ) is 1 if p is incident with ℓ and 0 otherwise.

- A *partial linear space* is an incidence structure such that each pair of distinct points are both incident with at most one line.

- A *partial linear space* is an incidence structure such that each pair of distinct points are both incident with at most one line.
- The point graph Γ of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the graph with vertex set $\mathcal{P}$ and two distinct points are adjacent if they are on a common line. Note that lines are cliques in Γ.

A result of Metsch

Metsch gave a sufficient condition for a DRG to be the point graph of a partial linear space.

A result of Metsch

Metsch gave a sufficient condition for a DRG to be the point graph of a partial linear space.

Theorem

Let Γ be a DRG. Assume that there exists a positive integer s such that the following two conditions are satisfied:

- $(s+1)(a_1+1) - k > (c_2 - 1)\binom{s+1}{2}$;
- $a_1 + 1 > (c_2 - 1)(2s - 1)$.

A result of Metsch

Metsch gave a sufficient condition for a DRG to be the point graph of a partial linear space.

Theorem

Let Γ be a DRG. Assume that there exists a positive integer s such that the following two conditions are satisfied:

- $(s+1)(a_1+1)-k > (c_2-1)\binom{s+1}{2}$;
- $a_1+1 > (c_2-1)(2s-1)$.

Define a line as a maximal clique with at least $a_1+2-(c_2-1)(s-1)$ vertices. Then $X=(V(\Gamma),\mathcal{L},\in)$ is a partial linear space, where $\mathcal{L}$ is the set of all lines, and Γ is the point graph of X. Moreover, every vertex is in at most s lines.

- For a DRG with classical parameters (D, b, α, β) with $b \geq 2$ and $D \geq 3$, the result of Metsch means that if $\beta > b^{D+5}$, then the graph is the point graph of partial linear space with large lines with $s \leq \frac{3}{2}b^{D}$.

- For a DRG with classical parameters (D, b, α, β) with $b \geq 2$ and $D \geq 3$, the result of Metsch means that if $\beta > b^{D+5}$, then the graph is the point graph of partial linear space with large lines with $s \leq \frac{3}{2}b^{D}$.
- Although a twisted Grassmann graph is the point graph of a partial linear space, its lines are not the maximum cliques. The twisted Grassmann graphs have $\beta > b^{D}$.

Geometric DRG

We first need the Delsarte bound for cliques for a DRG with classical parameters.

Geometric DRG

We first need the Delsarte bound for cliques for a DRG with classical parameters.

Lemma

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) with $D \geq 3$ and $b \geq 2$. Then the order c of a clique C in Γ is bounded by $c \leq \beta + 1$. If equality holds, the number of neighbours in C of a vertex not in C is $1 + \alpha$ or 0.

Geometric DRG

We first need the Delsarte bound for cliques for a DRG with classical parameters.

Lemma

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) with $D \geq 3$ and $b \geq 2$. Then the order c of a clique C in Γ is bounded by $c \leq \beta + 1$. If equality holds, the number of neighbours in C of a vertex not in C is $1 + \alpha$ or 0.

- A clique with equality in the lemma is called a *Delsarte clique*.

Geometric DRG

We first need the Delsarte bound for cliques for a DRG with classical parameters.

Lemma

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) with $D \geq 3$ and $b \geq 2$. Then the order c of a clique C in Γ is bounded by $c \leq \beta + 1$. If equality holds, the number of neighbours in C of a vertex not in C is $1 + \alpha$ or 0.

- A clique with equality in the lemma is called a *Delsarte clique*.
- A DRG Γ is called *geometric* if it is the point graph of a partial linear space with Delsarte cliques as its lines. This is equivalent with the condition that we can partition the edge set of Γ into Delsarte cliques.

Using the method of Metsch, he used in his bilinear forms graph paper in 1999, with some modifications and simplifications, we were able to show:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$.

Using the method of Metsch, he used in his bilinear forms graph paper in 1999, with some modifications and simplifications, we were able to show:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then there exists a constant $C_1 = C_1(\alpha, b)$ such that if $\beta \geq C_1 b^D$, then Γ is geometric. In particular $0 \leq \alpha \leq b$ is an integer.

Using the method of Metsch, he used in his bilinear forms graph paper in 1999, with some modifications and simplifications, we were able to show:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b \geq 2$. Then there exists a constant $C_1 = C_1(\alpha, b)$ such that if $\beta \geq C_1 b^D$, then Γ is geometric. In particular $0 \leq \alpha \leq b$ is an integer.

The C_1 is something like b^6 and the twisted Grassmann graphs are not geometric.

- With some extra work we were able to show that the local graphs are the α-clique-extension of a $\frac{\beta}{\alpha} \times \frac{b^D-1}{b-1}$-grid when $\alpha \neq 0$.
- The s-clique-extension of a graph G is replacing each vertex x by a clique C_x of order s and if $x \sim y$ in G then all the vertices of C_x are adjacent to all vertices of C_y.

- With some extra work we were able to show that the local graphs are the α-clique-extension of a $\frac{\beta}{\alpha} \times \frac{b^D-1}{b-1}$-grid when $\alpha \neq 0$.
- The s-clique-extension of a graph G is replacing each vertex x by a clique C_x of order s and if $x \sim y$ in G then all the vertices of C_x are adjacent to all vertices of C_y.
- For $\alpha = 0$ we were able to show that the local graphs are disjoint union of cliques of order β.

- With some extra work we were able to show that the local graphs are the α-clique-extension of a $\frac{\beta}{\alpha} \times \frac{b^D-1}{b-1}$-grid when $\alpha \neq 0$.
- The s-clique-extension of a graph G is replacing each vertex x by a clique C_x of order s and if $x \sim y$ in G then all the vertices of C_x are adjacent to all vertices of C_y.
- For $\alpha = 0$ we were able to show that the local graphs are disjoint union of cliques of order β.
- This leads to the question whether it is possible to classify the geometric DRG which are locally the clique extension of a grid.
- This seems to be a difficult problem.

Outline

- Let $\Gamma = (V, E)$ be a graph.
- A set π of subsets of V, $\pi = \{C_1, \ldots, C_t\}$ is called a partition of V if

$$\forall_i \; C_i \neq \emptyset,$$
$$\cup_1^t C_i = V$$
$$C_i \cap C_j \neq \emptyset \text{ if and only if } i = j.$$

- Let $\Gamma = (V, E)$ be a graph.
- A set π of subsets of V, $\pi = \{C_1, \ldots, C_t\}$ is called a partition of V if

  $\forall_i \; C_i \neq \emptyset$,
  $\cup_1^t C_i = V$
  $C_i \cap C_j \neq \emptyset$ if and only if $i = j$.
- A partition π of V is called *equitable* if there exist numbers β_{ij} $(1 \leq i, j \leq t)$ such that any $x \in C_i$ has exactly β_{ij} neighbours in C_j.
- We denote by $B = (\beta_{ij})$ the *quotient matrix* of π.

- We say a connected graph Γ has the 1*-homogeneous* property if, for every pair of vertices *x* and *y* at distance 1, the partition of the vertex set of Γ according to the path-length distance to both *x* and *y* is equitable, and the parameters corresponding to equitable partitions are independent of the choice of *x* and *y*.

- We say a connected graph Γ has the 1*-homogeneous* property if, for every pair of vertices x and y at distance 1, the partition of the vertex set of Γ according to the path-length distance to both x and y is equitable, and the parameters corresponding to equitable partitions are independent of the choice of x and y.
- Graphs with the 1-homogeneous property are simply said to be 1-homogeneous.

- We say a connected graph Γ has the 1*-homogeneous* property if, for every pair of vertices x and y at distance 1, the partition of the vertex set of Γ according to the path-length distance to both x and y is equitable, and the parameters corresponding to equitable partitions are independent of the choice of x and y.
- Graphs with the 1-homogeneous property are simply said to be 1-homogeneous.
- Note that a 1-homogeneous graph is always DRG.

Outline

- Let Γ be a distance-regular graph with diameter D.
- For integers s and t, we say that Γ is *of order* (s, t) if it is locally the disjoint union of $t + 1$ cliques of size s.

- Let Γ be a distance-regular graph with diameter D.
- For integers s and t, we say that Γ is *of order* (s, t) if it is locally the disjoint union of $t+1$ cliques of size s.
- A distance-regular graph Γ of order (s, t) is called a *regular near polygon* if $a_i = c_i a_1$ for all $1 \leq i \leq D-1$.

- Let Γ be a distance-regular graph with diameter D.
- For integers s and t, we say that Γ is *of order* (s, t) if it is locally the disjoint union of $t + 1$ cliques of size s.
- A distance-regular graph Γ of order (s, t) is called a *regular near polygon* if $a_i = c_i a_1$ for all $1 \leq i \leq D - 1$.
- If $a_D = c_D a_1$, we call Γ a *regular near* $2D$*-gon*; otherwise it is called a *regular near* $(2D + 1)$*-gon*.

- Let Γ be a distance-regular graph with diameter D.
- For integers s and t, we say that Γ is *of order* (s, t) if it is locally the disjoint union of $t+1$ cliques of size s.
- A distance-regular graph Γ of order (s, t) is called a *regular near polygon* if $a_i = c_i a_1$ for all $1 \leq i \leq D-1$.
- If $a_D = c_D a_1$, we call Γ a *regular near* $2D$*-gon*; otherwise it is called a *regular near* $(2D+1)$*-gon*.

Nomura 1990's showed the following result:

Lemma

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 2$ and $a_1 \geqslant 1$. Then Γ is locally disconnected if and only if Γ is a regular near $2D$-gon.

The following result gives a classification of the regular near $2D$-gons if $D \geq 4$. It is based on the work of many people including Brouwer, Wilbrink, Cohen, Cameron, De Bruyn, and so on. I already mentioned it before.

Theorem

Let G be a regular near $2D$-gon with $D \geqslant 4$ and $a_1 \geq 1$. If $c_2 \geqslant 3$ or $c_i = i$ $(i = 2, 3)$, then G is either a dual polar graph or a Hamming graph.

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near $2D$-gon;*

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near 2D-gon;*
- *Γ is a Johnson graph $J(2D, D)$;*

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near $2D$-gon;*
- *Γ is a Johnson graph $J(2D, D)$;*
- *Γ is a halved ℓ-cube where $\ell \in \{2D, 2D+1\}$;*

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near $2D$-gon;*
- *Γ is a Johnson graph $J(2D, D)$;*
- *Γ is a halved ℓ-cube where $\ell \in \{2D, 2D + 1\}$;*
- *Γ is a folded Johnson graph $\bar{J}(4D, 2D)$;*

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near $2D$-gon;*
- *Γ is a Johnson graph $J(2D, D)$;*
- *Γ is a halved ℓ-cube where $\ell \in \{2D, 2D + 1\}$;*
- *Γ is a folded Johnson graph $\bar{J}(4D, 2D)$;*
- *Γ is a folded halved $(4D)$-cube.*

K., B. Gebremichel, M. Abdullah, J.H. Lee (2024+) showed.

Theorem

Let Γ be a 1-homogeneous distance-regular graph with diameter $D \geqslant 5$ and $a_1 \geqslant 1$. Define $b = b_1/(\theta_1 + 1)$. Then, either $c_2 = 1$, or $b \geqslant 1$ and one of the following holds:

- *Γ is a regular near $2D$-gon;*
- *Γ is a Johnson graph $J(2D, D)$;*
- *Γ is a halved ℓ-cube where $\ell \in \{2D, 2D+1\}$;*
- *Γ is a folded Johnson graph $\bar{J}(4D, 2D)$;*
- *Γ is a folded halved $(4D)$-cube.*
- *The valency k of Γ is bounded by a function $F(b)$ of b,* i.e., *$k \leqslant F(b)$, where*
  $F(b) = 16(b+1)^{10} + O((b+1)^9)$, and $b \geq 2$.

The proof consists of three parts:

- Let $b \geq 1$ be fixed.
- First by using a result of Neumaier we may assume that the local graph is a Latin square graph or a Steiner graph, if the valency is large enough.

The proof consists of three parts:

- Let $b \geq 1$ be fixed.
- First by using a result of Neumaier we may assume that the local graph is a Latin square graph or a Steiner graph, if the valency is large enough.
- Next: If $c_2 \in \{(b+1)^2, b(b+1)\}$ then it is impossible.

The proof consists of three parts:

- Let $b \geq 1$ be fixed.
- First by using a result of Neumaier we may assume that the local graph is a Latin square graph or a Steiner graph, if the valency is large enough.
- Next: If $c_2 \in \{(b+1)^2, b(b+1)\}$ then it is impossible.
- Last step. Show that if the valency is large enough, then $c_2 \in \{(b+1)^2, b(b+1)\}$. In the talk by Dr. Gebremichel, he will discuss this part.

Outline

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.
- First we need some definitions.
- A regular graph is called **co-edge-regular** (with parameter c) if any two distinct non-adjacent vertices have exactly c common neighbours.

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.
- First we need some definitions.
- A regular graph is called **co-edge-regular** (with parameter *c*) if any two distinct non-adjacent vertices have exactly *c* common neighbours.
- We call a DRG Γ μ**-graph-regular** if there exists a *t* such that the subgraph induced on the common neighbours of two vertices *x* and *y* at distance 2 is *t*-regular, not depending on the pair x, y.

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.
- First we need some definitions.
- A regular graph is called **co-edge-regular** (with parameter c) if any two distinct non-adjacent vertices have exactly c common neighbours.
- We call a DRG Γ μ-**graph-regular** if there exists a t such that the subgraph induced on the common neighbours of two vertices x and y at distance 2 is t-regular, not depending on the pair x, y.
- If the DRG is μ-graph-regular, then each local graph is a co-edge-regular graph with parameter t.

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.
- First we need some definitions.
- A regular graph is called **co-edge-regular** (with parameter c) if any two distinct non-adjacent vertices have exactly c common neighbours.
- We call a DRG Γ **μ-graph-regular** if there exists a t such that the subgraph induced on the common neighbours of two vertices x and y at distance 2 is t-regular, not depending on the pair x, y.
- If the DRG is μ-graph-regular, then each local graph is a co-edge-regular graph with parameter t.
- Terwilliger (1990's) showed that a thin DRG with classical parameters is μ-graph-regular, if the diameter is at least 5.

- In this section we will study thin DRG with classical parameters (D, b, α, β), with $b \geq 2$.
- First we need some definitions.
- A regular graph is called **co-edge-regular** (with parameter c) if any two distinct non-adjacent vertices have exactly c common neighbours.
- We call a DRG Γ **μ-graph-regular** if there exists a t such that the subgraph induced on the common neighbours of two vertices x and y at distance 2 is t-regular, not depending on the pair x, y.
- If the DRG is μ-graph-regular, then each local graph is a co-edge-regular graph with parameter t.
- Terwilliger (1990's) showed that a thin DRG with classical parameters is μ-graph-regular, if the diameter is at least 5.
- The bilinear forms graphs are non-thin DRG with classical parameters, but they are still μ-graph-regular.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.
- Assume that Γ is thin.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.
- Assume that Γ is thin.
- Then Terwilliger (1990's) showed that the nontrivial eigenvalues of $\Delta(x)$ are in $\{\beta - \alpha - 1, \alpha b \frac{b^{D-1}-1}{b-1} - 1, -1, -b-1\}$.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.
- Assume that Γ is thin.
- Then Terwilliger (1990's) showed that the nontrivial eigenvalues of $\Delta(x)$ are in $\{\beta - \alpha - 1, \alpha b \frac{b^{D-1} - 1}{b - 1} - 1, -1, -b - 1\}$.
- Let us assume that $a_1 > 0$, otherwise Γ is bipartite.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.
- Assume that Γ is thin.
- Then Terwilliger (1990's) showed that the nontrivial eigenvalues of $\Delta(x)$ are in $\{\beta - \alpha - 1, \alpha b \frac{b^{D-1}-1}{b-1} - 1, -1, -b-1\}$.
- Let us assume that $a_1 > 0$, otherwise Γ is bipartite.
- It follows by results of Curtin and Nomura (2000's) that if $\Delta(x)$ has only two distinct non-trivial eigenvalues, then Γ is 1-homogeneous.

- Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 3$, $b \geq 1$.
- Let $\Delta(x)$ be the local graph at vertex x.
- Assume that Γ is thin.
- Then Terwilliger (1990's) showed that the nontrivial eigenvalues of $\Delta(x)$ are in $\{\beta - \alpha - 1, \alpha b \frac{b^{D-1}-1}{b-1} - 1, -1, -b-1\}$.
- Let us assume that $a_1 > 0$, otherwise Γ is bipartite.
- It follows by results of Curtin and Nomura (2000's) that if $\Delta(x)$ has only two distinct non-trivial eigenvalues, then Γ is 1-homogeneous.
- In this case, we obtain that Γ is a Hamming graph, a Jonson graph, a halved cube, a dual polar graph, or $D \leq 9$, by the results on 1-homogeneous DRG.

With H. Ge (ongoing work) we obtained the following two results:

With H. Ge (ongoing work) we obtained the following two results:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 5$, $b \geq 1$. If $\alpha = 0$ and Γ is thin, then Γ is a dual polar graph or Hamming graph.

With H. Ge (ongoing work) we obtained the following two results:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 5$, $b \geq 1$. If $\alpha = 0$ and Γ is thin, then Γ is a dual polar graph or Hamming graph.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b = 2$. Assume that Γ is thin and that $\Delta(x)$ has exactly 4 distinct eigenvalues. Then Γ is the Grassmann graph $J_2(2D, D)$.

With H. Ge (ongoing work) we obtained the following two results:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 5$, $b \geq 1$. If $\alpha = 0$ and Γ is thin, then Γ is a dual polar graph or Hamming graph.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b = 2$. Assume that Γ is thin and that $\Delta(x)$ has exactly 4 distinct eigenvalues. Then Γ is the Grassmann graph $J_2(2D, D)$.

Conjecture

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b = 2$. Assume that Γ is thin. Then Γ is a Grassmann graph $J_2(n, D)$ with $n \geq 2D$, or a dual polar graph.

With H. Ge (ongoing work) we obtained the following two results:

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 5$, $b \geq 1$. If $\alpha = 0$ and Γ is thin, then Γ is a dual polar graph or Hamming graph.

Theorem

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b = 2$. Assume that Γ is thin and that $\Delta(x)$ has exactly 4 distinct eigenvalues. Then Γ is the Grassmann graph $J_2(2D, D)$.

Conjecture

Let Γ be a DRG with classical parameters (D, b, α, β) such that $D \geq 9$, and $b = 2$. Assume that Γ is thin. Then Γ is a Grassmann graph $J_2(n, D)$ with $n \geq 2D$, or a dual polar graph.

Thank you for your attention.