Minimum number of disjoint pairs in a uniform family of subsets

Gyula O.H. Katona
Rényi Institute, Budapest

G2C2
Hebei Normal Universíty

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Erdős-Ko-Rado theorem

Let $[n] = \{1, 2, \ldots, n\}$ be our underlying set. $\binom{[n]}{k}$ will denote the family of all k-element subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called intersecting if any pair of its members has a non-empty intersection.

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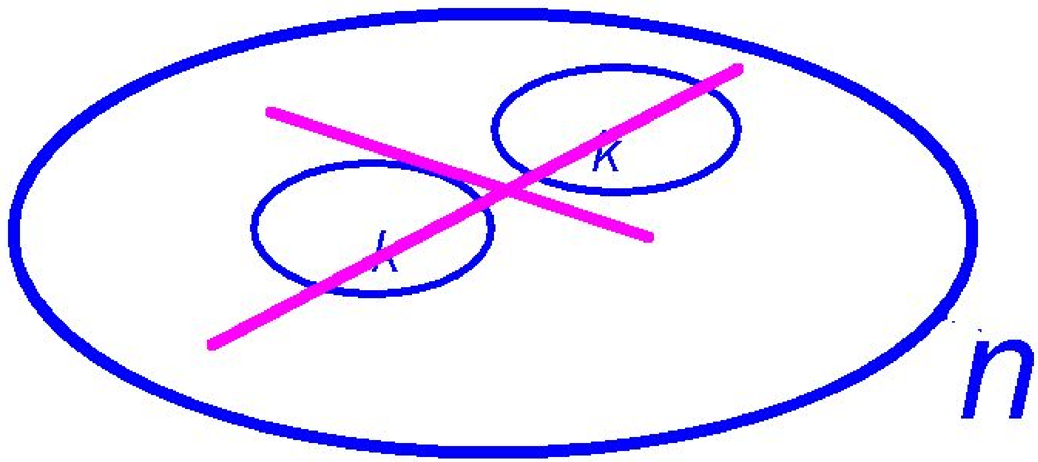

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Theorem (Erdős-Ko-Rado, 1961) If $2k \le n, \mathcal{F} \subset \binom{[n]}{k}$ is intersecting then

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

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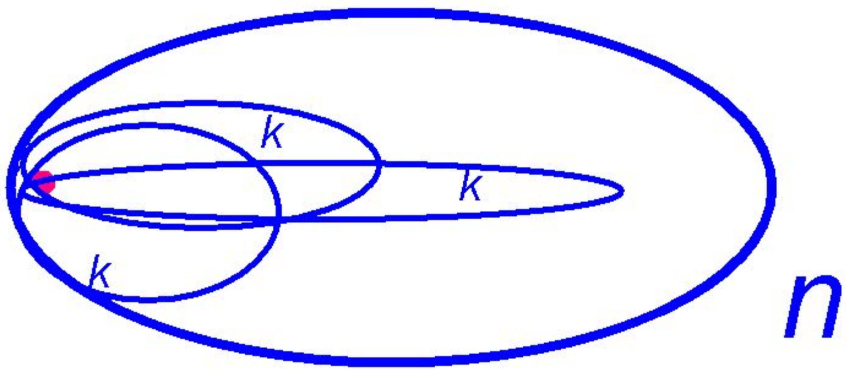

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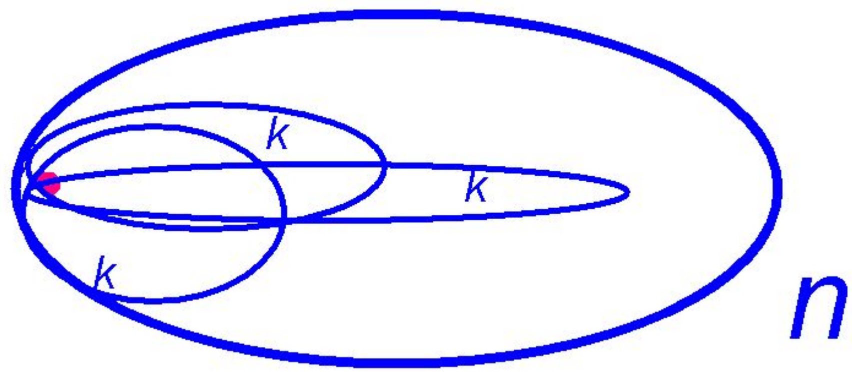

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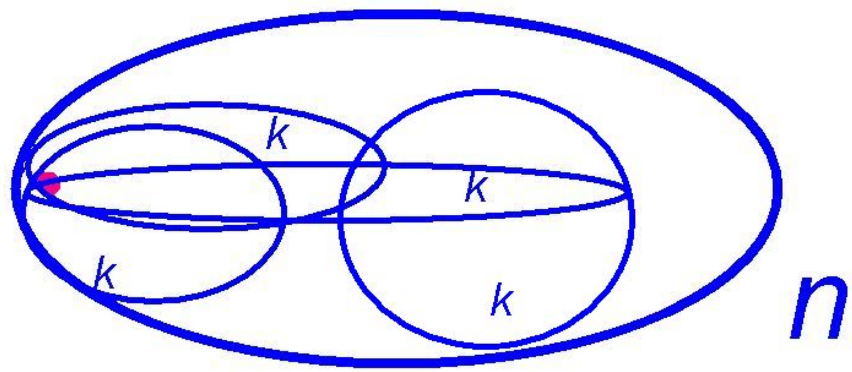

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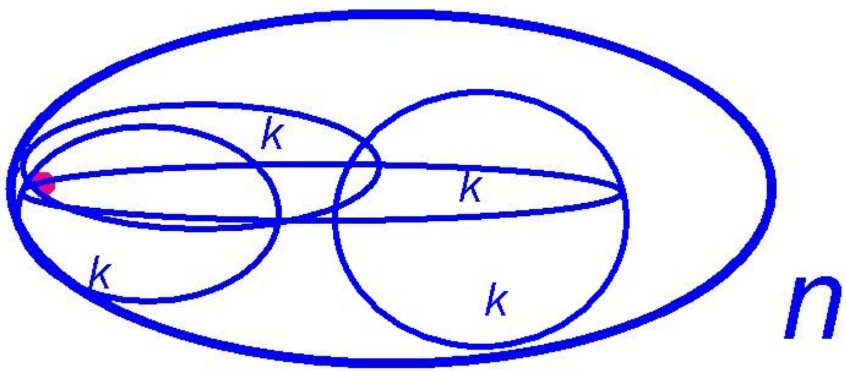
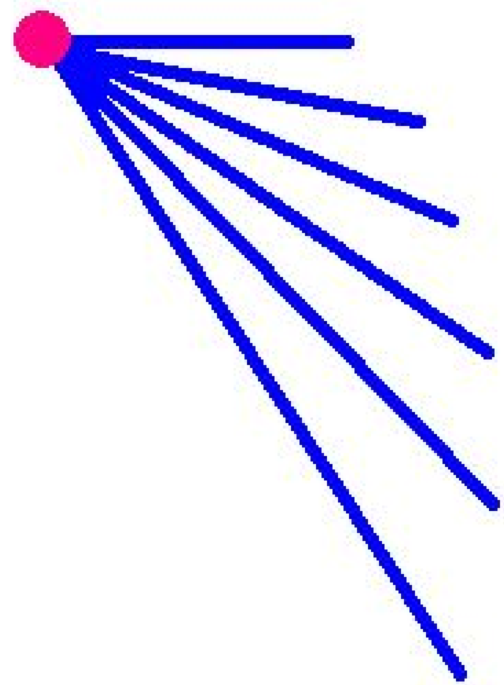


Theorem (Gyula O.H. Katona, Gyula Y. Katona, Zs. Katona, 2012) If $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$ then there are at least $\binom{n-k-1}{k-1}$ disjoint pairs.

k=2



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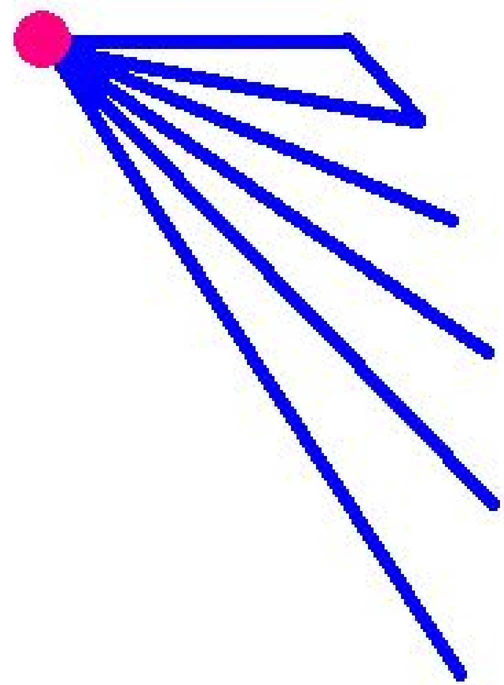

lexicographic ordering of k-element subsets

characteristic vector of the set $A \subset [n]$

(0,	0,	1,	1,	0,	...	1,	...,	0)
1	2	3	4	5	...	k	...	n

the uth coordinate is 1 iff $u \in A$

lexicographic ordering of k-element subsets

$n = 5, k = 3$, $|\mathcal{F}| = \binom{5}{3} = 10$

(0, 0, 1, 1, 1)
(0, 1, 0, 1, 1)
(0, 1, 1, 0, 1)
(0, 1, 1, 1, 0)
(1, 0, 0, 1, 1)
(1, 0, 1, 0, 1)
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$k = 2$, **EKR**

$n = 5, k = 2$

(1,	1,	0,	0,	0)
(1,	0,	1,	0,	0)
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$$\begin{array}{ccccc}(1, & 1, & 0, & 0, & 0)\\(1, & 0, & 1, & 0, & 0)\\(1, & 0, & 0, & 1, & 0)\\(1, & 0, & 0, & 0, & 1)\end{array}$$

last $n - 1$ sets in lexicographic order

$k = 2$, n sets

$n = 5, k = 2$, smallest number of disjoint pairs

(1,	1,	0,	0,	0)
(1,	0,	1,	0,	0)
(1,	0,	0,	1,	0)
(1,	0,	0,	0,	1)
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$k = 2$, $n + 1$ sets

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$$
\begin{array}{ccccc}
(1, & 1, & 0, & 0, & 0) \\
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(0, & 1, & 1, & 0, & 0) \\
(0, & 1, & 0, & 1, & 0)
\end{array}
$$

last $n + 1$ sets in lexicographic order

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$n = 5, k = 2$, smallest number of disjoint pairs=**4**?

$$\begin{matrix} (1, & 1, & 0, & 0, & 0) \\ (1, & 0, & 1, & 0, & 0) \\ (1, & 0, & 0, & 1, & 0) \\ (1, & 0, & 0, & 0, & 1) \\ (0, & 1, & 1, & 0, & 0) \\ (0, & 1, & 0, & 1, & 0) \end{matrix}$$

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first 6 sets in lexicographic order

$k = 2$, 6 sets

$n = 5, k = 2$, smallest number of disjoint pairs=**3**

(0,	0,	0,	1,	1)
(0,	0,	1,	0,	1)
(0,	0,	1,	1,	0)
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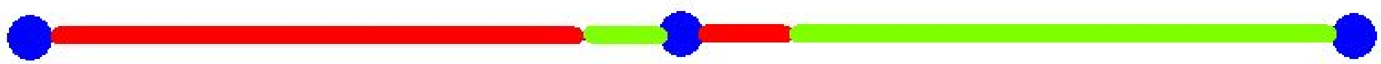

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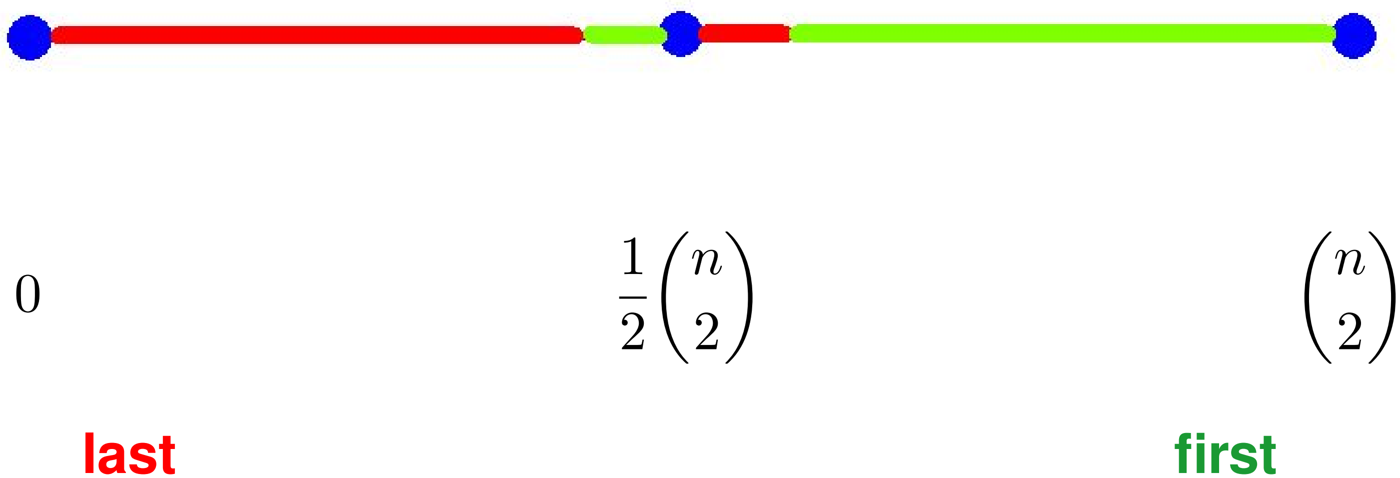

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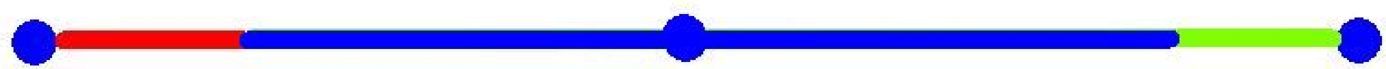

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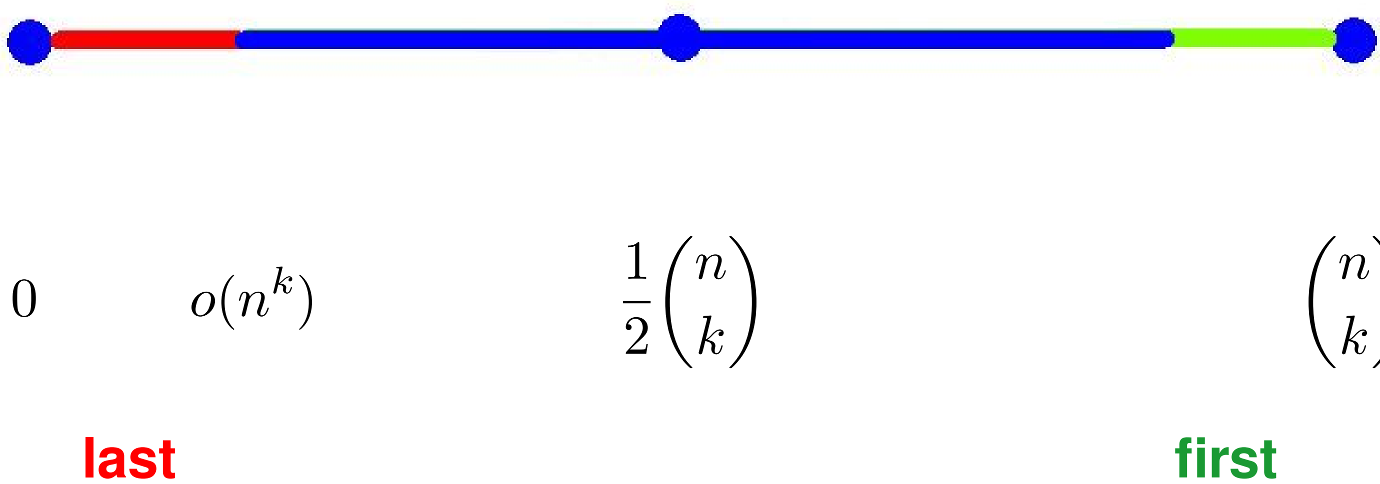
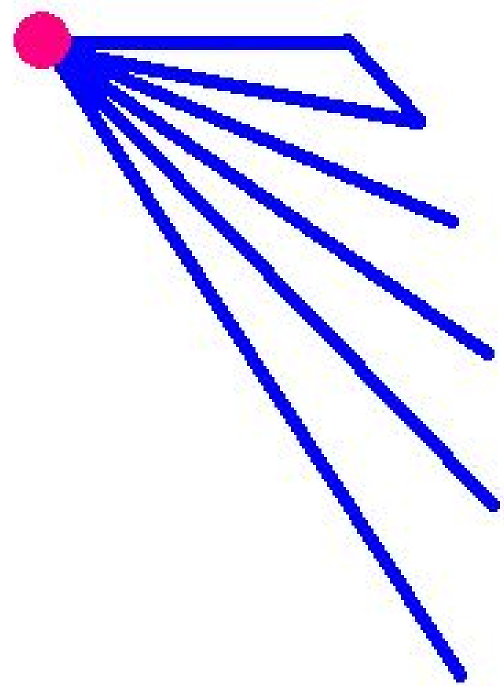


The structure of the disjoint pairs



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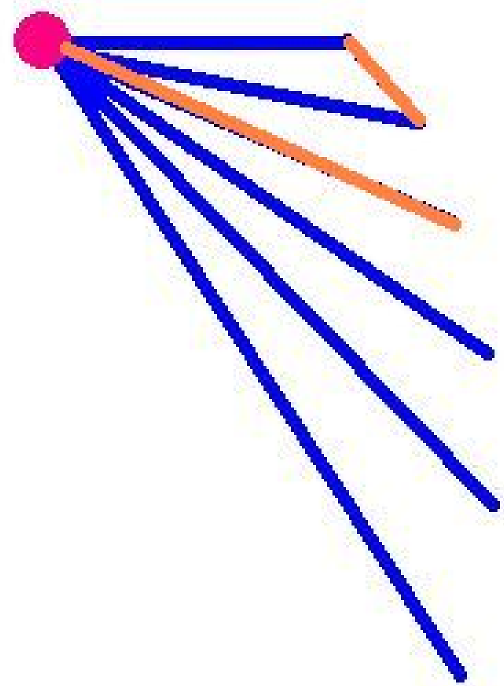

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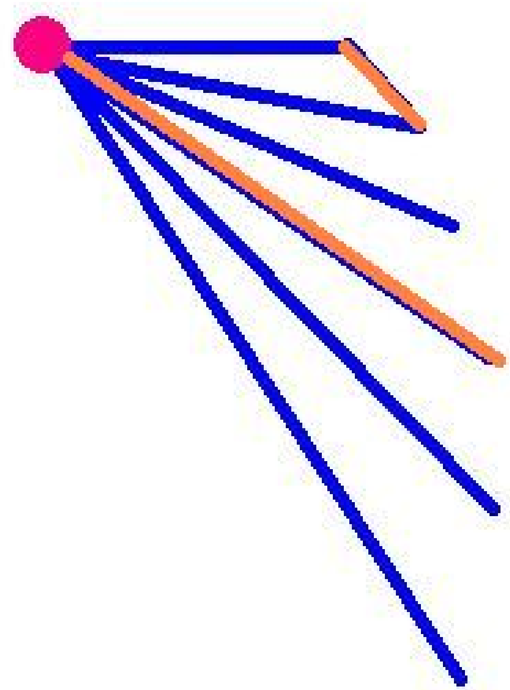

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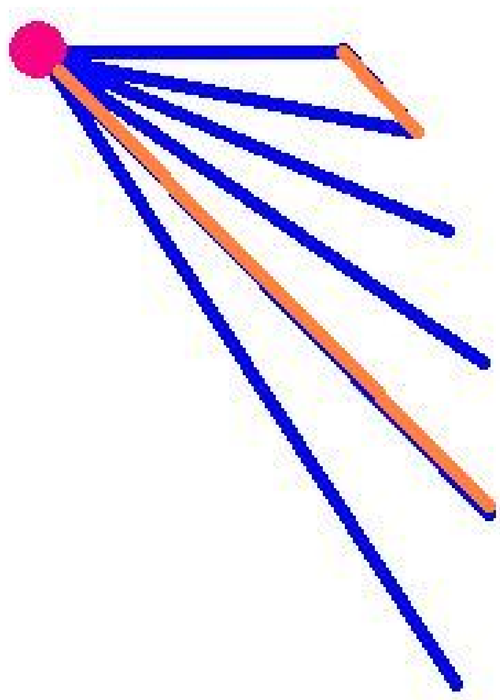

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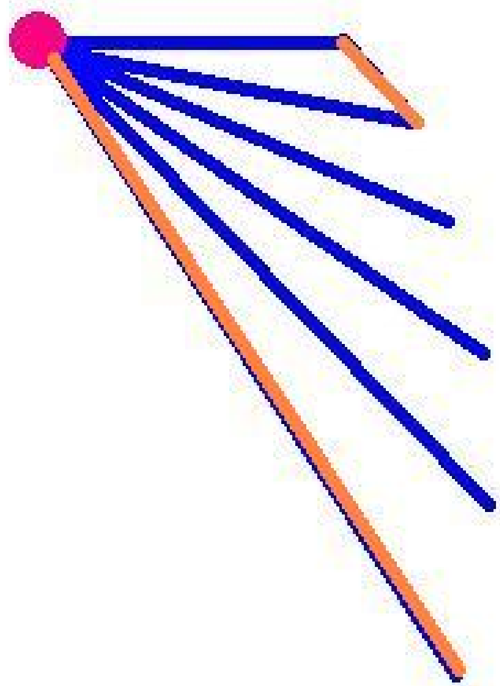

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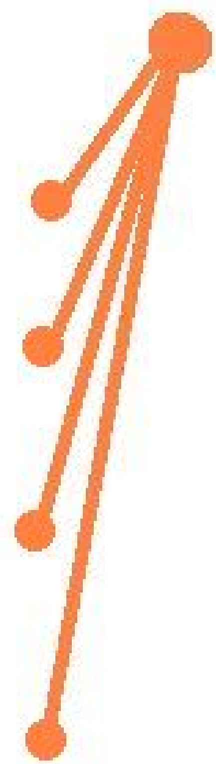

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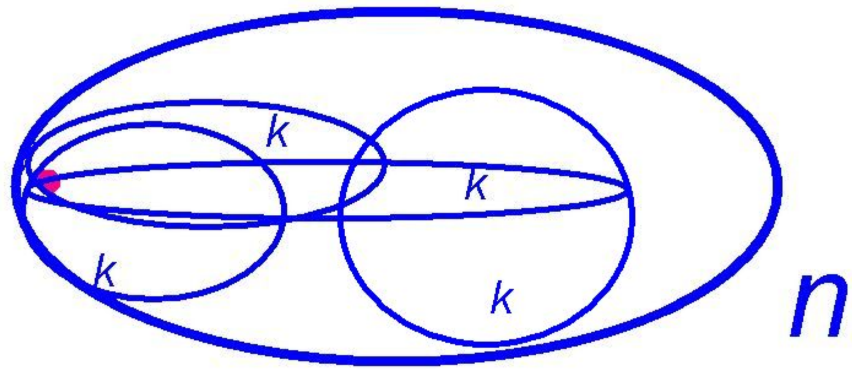

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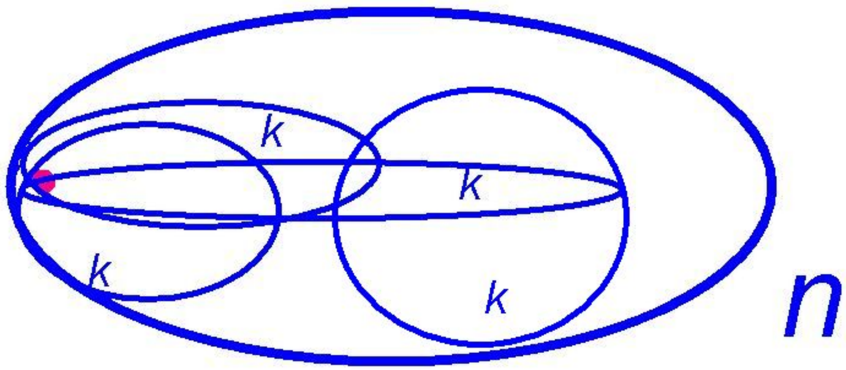

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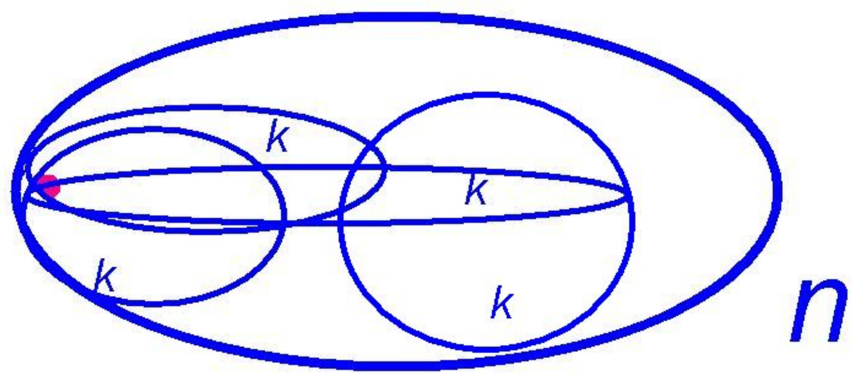

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Zsolt Katona

Haas School of Business
University of California, Berkeley

Professor
Marketing

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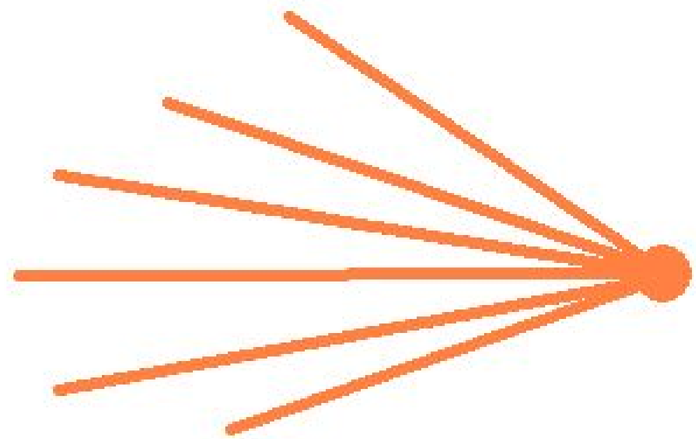

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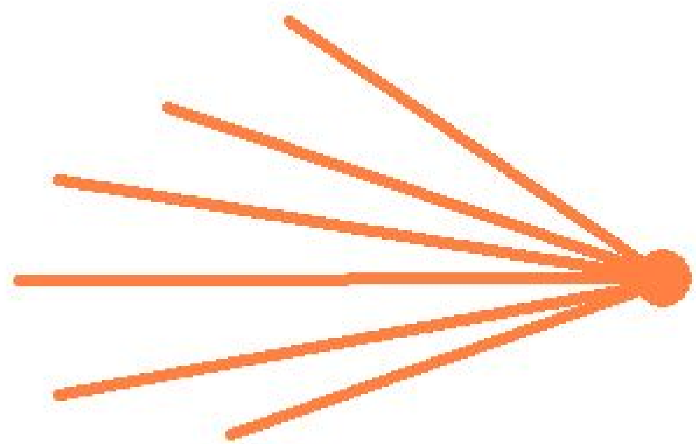

Star!

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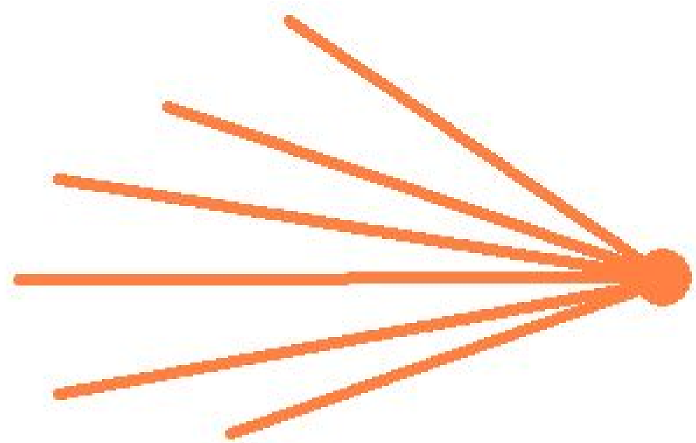

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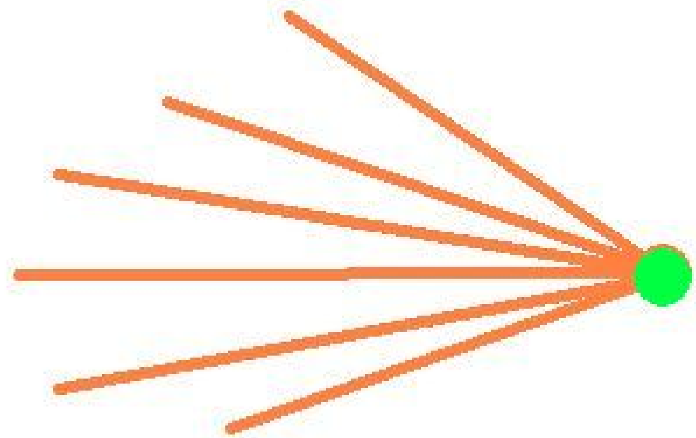

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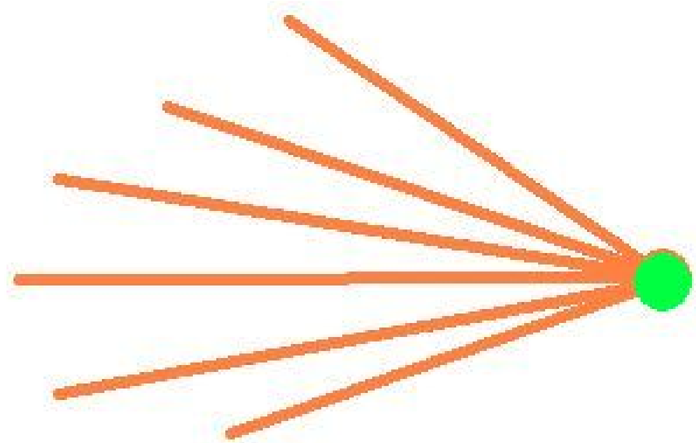

$\tau(G) = 1$ iff G is a star.

$$k = 2, \quad |\mathcal{F}| = \binom{n-1}{k-1} + 1 = n, \quad \tau(\mathrm{DP}(\mathcal{F})) > 1$$

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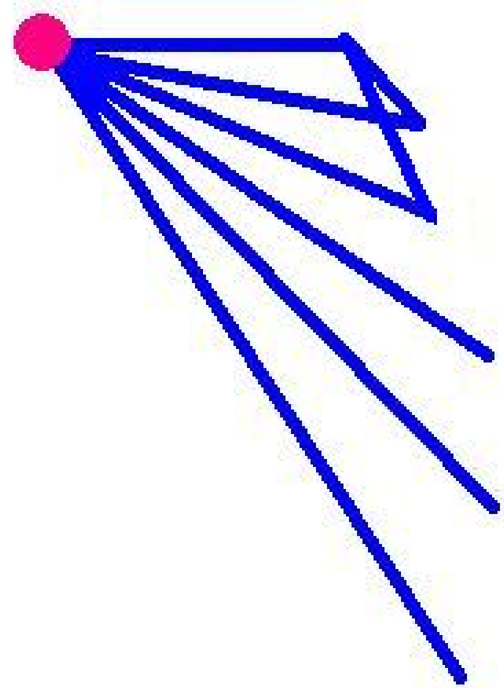

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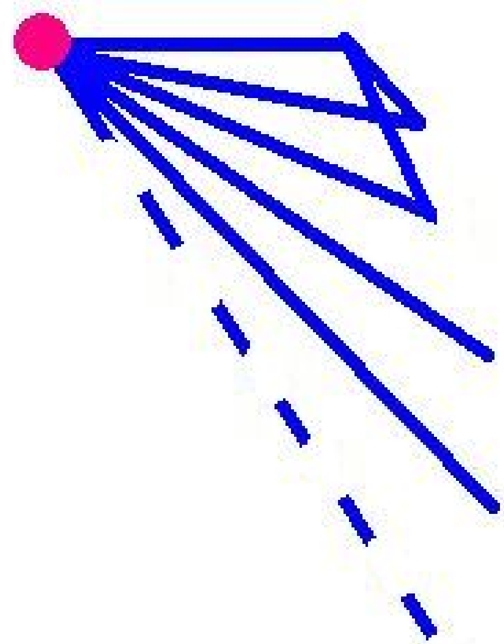

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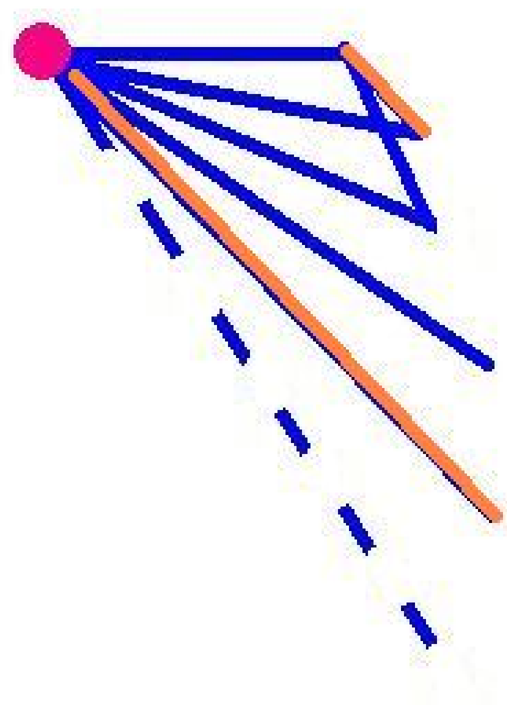

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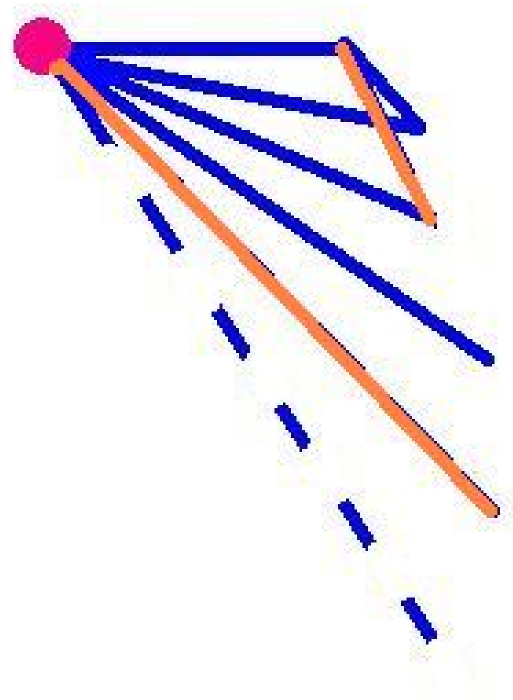

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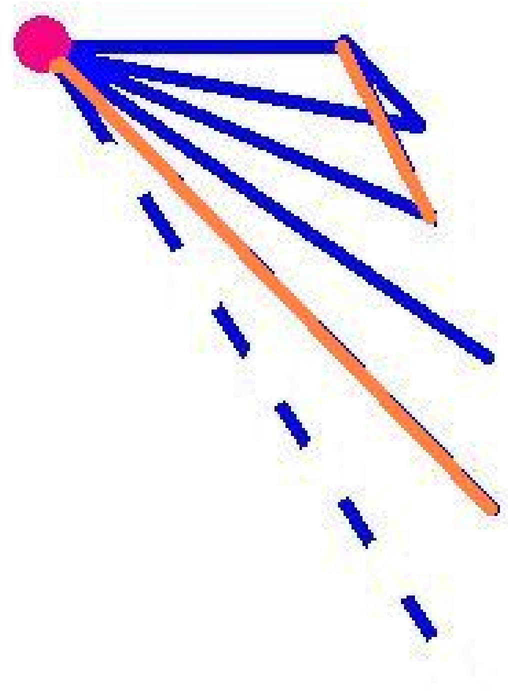

$\mathsf{dp}(\mathcal{F}) = 2(n-2)$

general k

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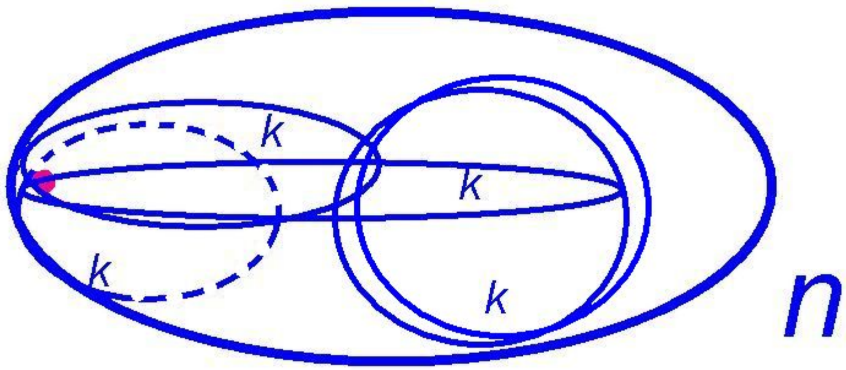

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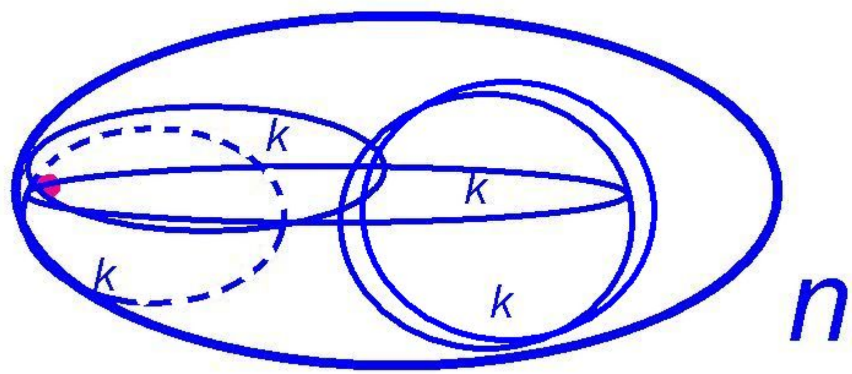

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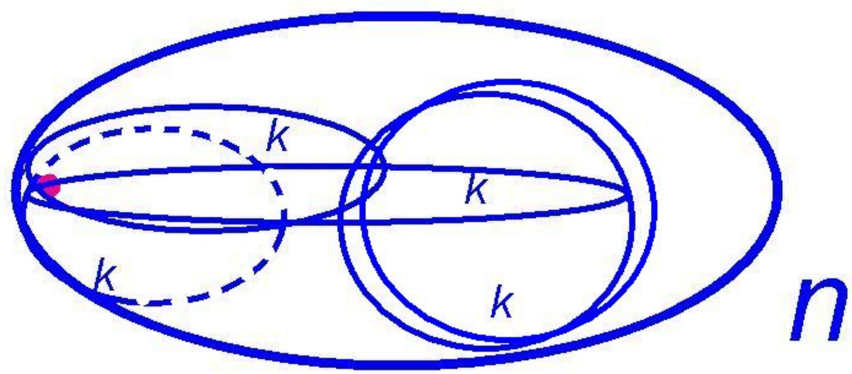
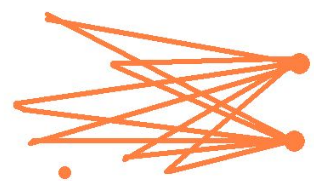


$\binom{n-k-1}{k-1} - 1$



$\mathsf{dp}(\mathcal{F}) = 2\left(\binom{n-k-1}{k-1} - 1\right)$

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Theorem **(Jasińska-Katona, 2024+)** Suppose that $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$. Then either $\mathcal{F}$ is a trivially intersecting family plus one more k-element set, or

$$\mathrm{dp}(\mathcal{F}) \geq 2\binom{n-k-1}{k-1} - 2$$

if n is large enough.

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if n is large enough. $\Longleftarrow$ **this should be removed**

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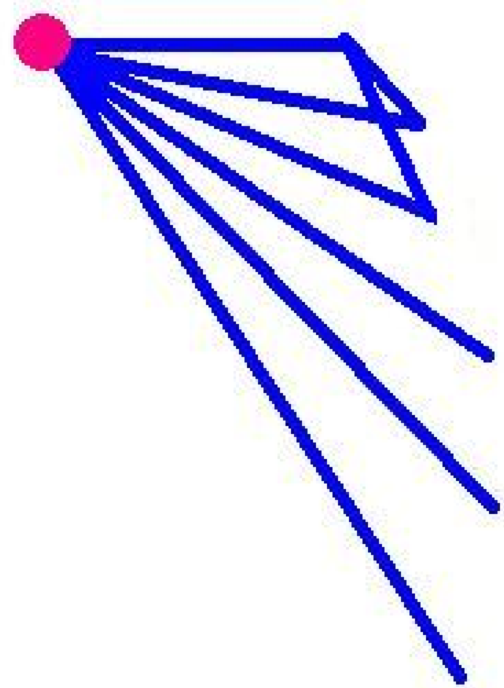

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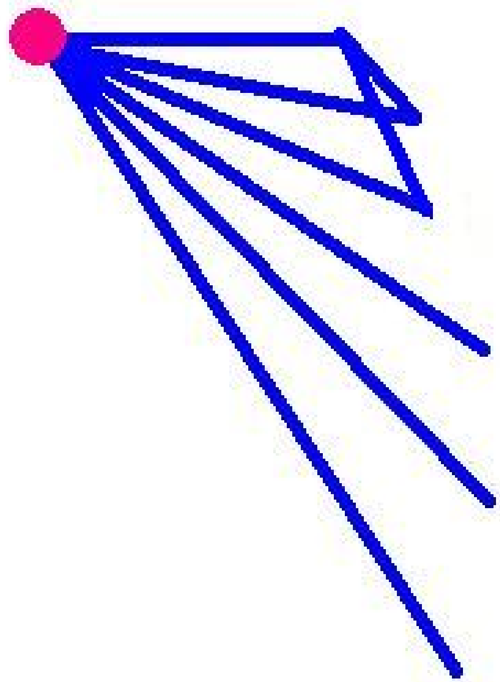

$$\mathrm{dp}(\mathcal{F}) = 2(n-3)$$

The general case

$$2k \leq n, \quad \mathcal{F} \subset \binom{[n]}{k}, \quad |\mathcal{F}| = \binom{n-1}{k-1} + r, \quad \tau(\mathrm{DP}(\mathcal{F})) \geq s$$

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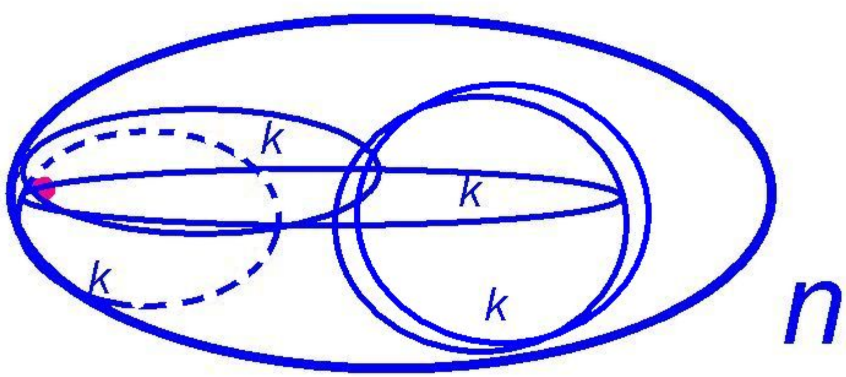
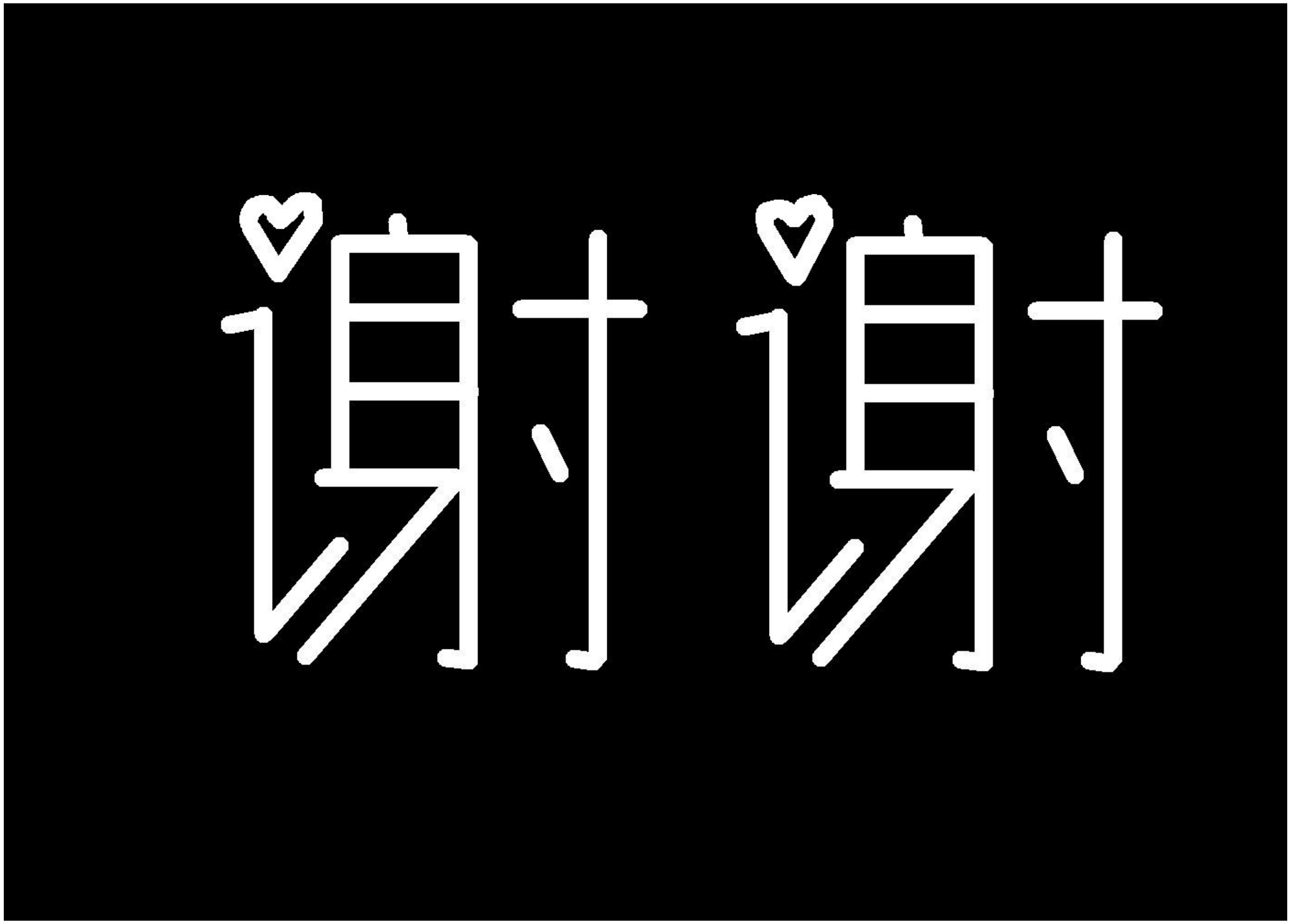


delete $s - r$ sets **add s sets**

谢谢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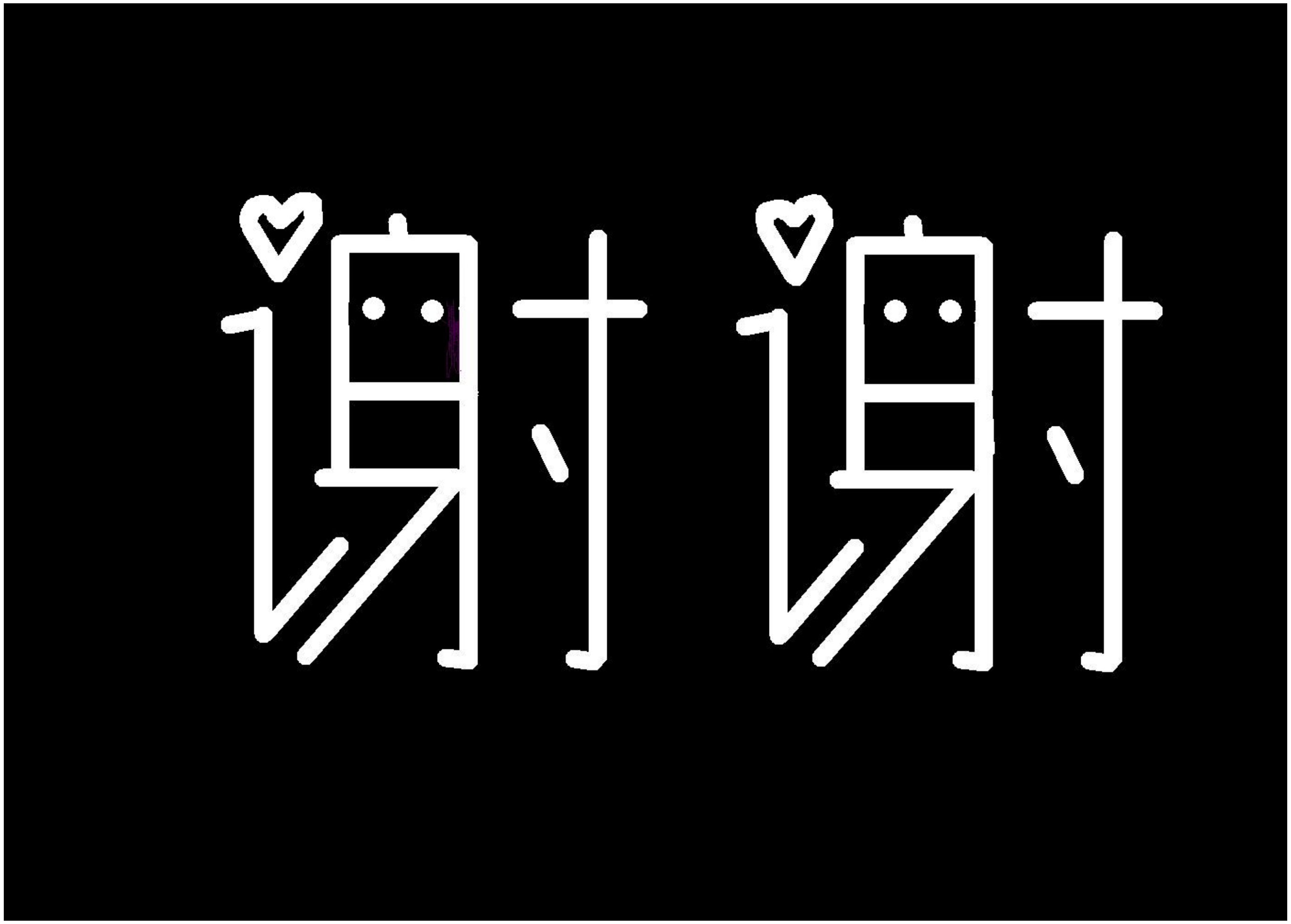
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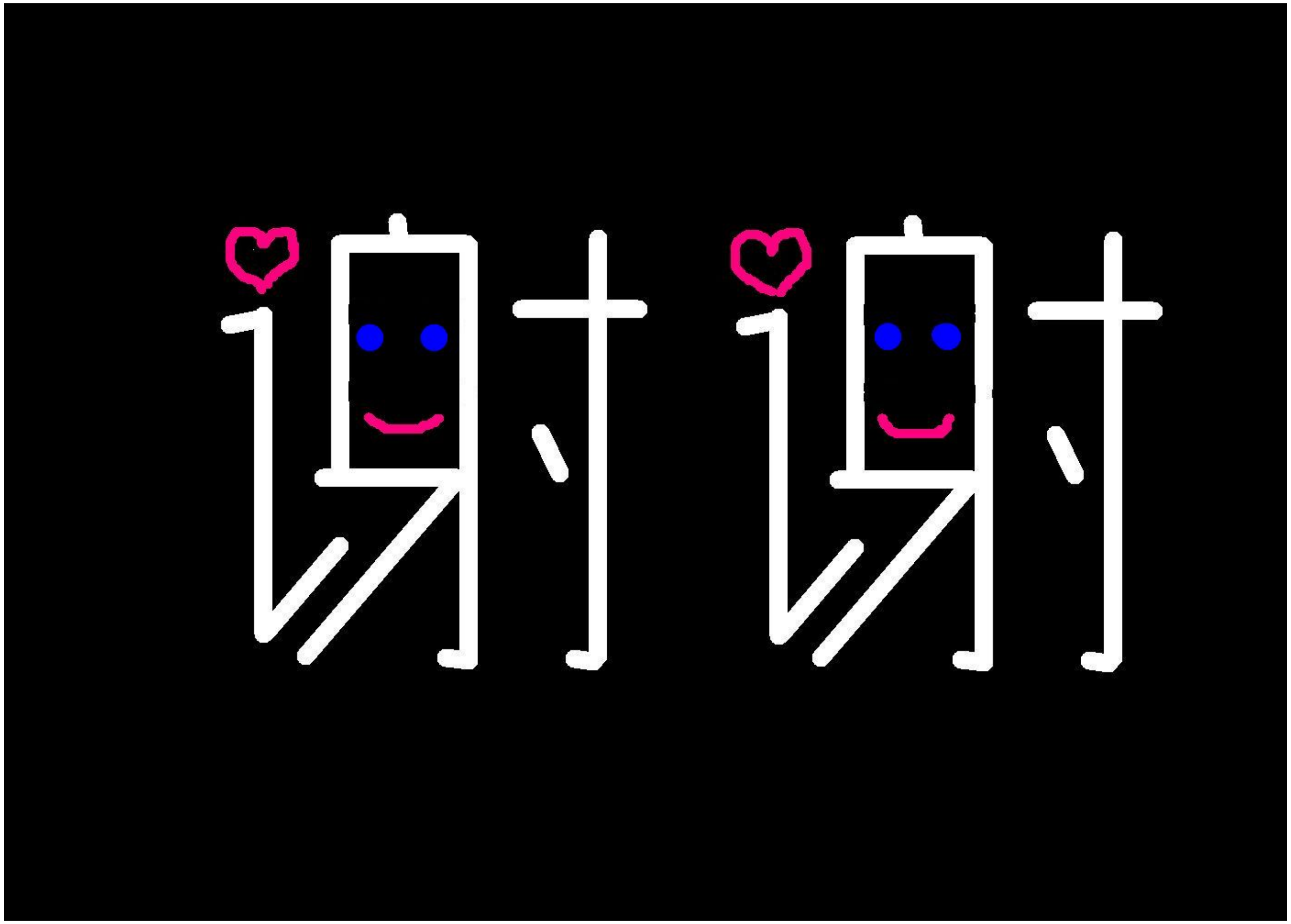
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Thank you for your attention!