First-order model theory, surjunctivity, and Kaplansky's stable finiteness conjecture

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This is joint work:

T.Ceccherini-Silberstein, M. Coornaert, and X.K. Phung:
"First-order model theory and Kaplansky's stable finiteness conjecture for surjunctive groups", arXiv:2310.09451, to appear in *Groups, Geometry, and Dynamics*.

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For a ring R, the following conditions are equivalent:

- DF1 R is directly finite;
- DF2 every left-invertible element in R is invertible;
- DF3 every right-invertible element in R is invertible;
- DF4 R is Hopfian as a left R-module;
- DF5 R is Hopfian as a right-module.

(A module M is called *Hopfian* if every surjective endomorphism of M is an automorphism.)

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For a ring R, the following conditions are equivalent:

- (SF1) R is stably finite;
- (SF2) $\forall d \geq 1, \forall A, B \in \mathrm{Mat}_d(R), \quad AB = I_d \implies BA = I_d$,
- (SF3) $\forall d \geq 1$, the left R-module R^d is Hopfian;
- (SF4) $\forall d \geq 1$, the right R-module R^d is Hopfian;
- (SF5) every finitely generated free left R-module is Hopfian;
- (SF6) every finitely generated free right R-module is Hopfian.

Examples of stably finite rings

- Any finite ring is stably finite.
- Any commutative ring is stably finite.
- Any field is stably finite.
- Any division ring is stably finite.
- Any left (or right) Noetherian ring is stably finite.
- If V is a vector space over a field K then $\mathrm{End}_K(V)$ is stably finite iff $\dim_K(V) < \infty$.
- Any unit-regular ring is stably finite.

Stable finiteness vs direct finiteness

Stable finiteness vs direct finiteness

R stably finite $\implies$ R directly finite (since $\mathrm{Mat}_1(R) = R$).
There exist directly finite rings that are not stably finite.
For any $d \geq 1$, there exist rings R such that $\mathrm{Mat}_d(R)$ is directly finite but $\mathrm{Mat}_{d+1}(R)$ is not.

Group algebras

Group algebras

Let G be a group and let K be a field.
The group algebra of G with coefficients in K is the K-algebra $K[G]$ constructed as follows:

- $K[G]$ is the K-vector space with base G;
- the multiplication on $K[G]$ is obtained by extending K-linearly the group operation on G.

Thus, every $\alpha \in K[G]$ can be uniquely written in the form

$$\alpha = \sum_{g \in G} \alpha_g g$$

with $\alpha_g \in K$ for all $g \in G$ and $\alpha_g = 0$ for all but finitely many $g \in G$.

Group algebras (continued)

The operations on $K[G]$ are given by the formulae:

$$\alpha + \beta = \sum_{g \in G} (\alpha_g + \beta_g) g,$$

$$\lambda \alpha = \sum_{g \in G} (\lambda \alpha_g) g,$$

$$\alpha \beta = \sum_{g \in G} \left(\sum_{h_1, h_2 \in G : h_1 h_2 = g} \alpha_{h_1} \beta_{h_2} \right) g$$

for all $\alpha, \beta \in K[G]$ and $\lambda \in K$.

Kaplansky's stable finiteness conjecture

Theorem (Kaplansky)

Let G be a group and let K be a field of characteristic 0. Then the group algebra $K[G]$ is stably finite.

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Kaplansky's stable finiteness conjecture: The group algebra $K[G]$ is stably finite for every group G and every field K.

Symbolic dynamics

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$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx := x \circ L_{g^{-1}} \end{aligned}$$

where $L_{g^{-1}} \colon G \to G$ is the left multiplication by g^{-1}.

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The prodiscrete topology on A^G is the product topology obtained by taking the discrete topology on every factor A of $A^G = \prod_{g \in G} A$. The G-shift on A^G is continuous. The space A^G is homeomorphic to the Cantor space for $|A| \geq 2$ and G countably infinite.

Surjunctive groups

The notion of a surjunctive group goes back to Gottschalk.

Definition

A group G is called surjunctive if, for any finite set A and every continuous G-equivariant map $\tau\colon A^G \to A^G$, one has

$$\tau \text{ injective} \implies \tau \text{ surjective}.$$

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$$\tau \text{ injective } \implies \tau \text{ surjective.}$$

No example of a non-surjunctive group has been found up to now.
Gottschalk's conjecture: Every group is surjunctive.

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No example of a non-sofic group has been found up to now.

Theorem (Gromov and Weiss)

Every sofic group is surjunctive.

Definition of sofic groups

Given a non-empty finite set X, let $\mathrm{Sym}(X)$ denote the symmetric group of X.

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Definition

A group G is called sofic if for every $\varepsilon > 0$ and for every finite subset $F \subset G$, there exist a non-empty finite set X and a map $\phi \colon F \to \mathrm{Sym}(X)$ such that

- $\forall g, h \in F, \quad gh \in F \implies d_X(\phi(gh), \phi(g)\phi(h)) \leq \varepsilon$;
- $\forall g, h \in F, \quad g \neq h \implies d_X(\phi(g), \phi(h)) \geq 1 - \varepsilon$.

Stable finiteness of group algebras of surjunctive groups

The following result was obtained by Xuan Kien Phung using algebraic geometry.

Theorem A (Phung)

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As sofic $\implies$ surjunctive by the Gromov-Weiss theorem, we get.

Corollary (Elek et Szabó)

Every sofic group satisfies Kaplansky's stable finiteness conjecture.

Elementary equivalent fields

Two fields are called elementary equivalent if they satisfy the same first-order sentences in the language of rings $L = \{+, -, \times, 0, 1\}$.

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- The sentence $\forall x, \exists y, x = y^3$ is satisfied by $\mathbb{R}$ but not by $\mathbb{Q}$. Thus, the fields $\mathbb{R}$ and $\mathbb{Q}$ are not elementary equivalent.

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- Two isomorphic fields are always elementary equivalent.
- If two fields are elementary equivalent then they have the same characteristic.

The Lefschetz principles

The two following results may be found in the monograph of Marker (David Marker, Model theory, vol. 217, Graduate Texts in Mathematics, An introduction, Springer-Verlag, New York, 2002).

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Example

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. The fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are elementary equivalent. Observe that the fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are not isomorphic since $\overline{\mathbb{Q}}$ is countable while $\mathbb{C}$ is uncountable.

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Theorem (Second Lefschetz principle)

Let ψ be a first-order sentence in the language of rings which is satisfied by some (and hence any) algebraically closed field of characteristic 0. Then there exists an integer N such that ψ is satisfied by every algebraically closed field of characteristic $p \geq N$.

Proof of Theorem A

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Lemma 1

Let G be a group, let $d \geq 1$ be an integer, and let S be a finite subset of G. Then there exists a first-order sentence ψ in the language of rings such that a field K satisfies ψ if and only if there exist matrices $A, B \in \mathrm{Mat}_d(K[G])$ such that

1. *the support of each entry of A and of each entry of B is contained in S;*
2. *$AB = I_d$ and $BA \neq I_d$.*

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1. *the support of each entry of A and of each entry of B is contained in S;*
2. *$AB = I_d$ and $BA \neq I_d$.*

Lemma 2

Let G be a group and suppose that K and L are elementary equivalent fields. Then $K[G]$ is stably finite if and only if $L[G]$ is stably finite.

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Case 1: the field K is finite Let $d \geq 1$. Set $A := K^d$.

Proof of Theorem A (continued)

Let G be a surjunctive group and let K be a field. We want to show that $K[G]$ is stably finite.

Case 1: the field K is finite Let $d \geq 1$. Set $A := K^d$. A result by TCS and Coornaert says that $\mathrm{Mat}_d(K[G])$ is directly finite if and only if every injective, K-linear, G-equivariant and continuous map $\tau \colon A^G \to A^G$ is surjective.

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Proof of Theorem A (continued)

Let G be a surjunctive group and let K be a field. We want to show that $K[G]$ is stably finite.

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Case 2: the field K is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime p

Proof of Theorem A (continued)

Case 2: the field K is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime p Consider the Frobenius automorphism $\phi\colon K \to K$ defined by

$$\forall \lambda \in K, \quad \phi(\lambda) := \lambda^p$$

For $n \geq 1$, define $K_n \subset K$ by

$$K_n := \mathrm{Fix}(\underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}).$$

Proof of theorem A (continued)

- K_n is the set of roots of the polynomial $X^{p^n} - X$.
- K_n is a subfield of K with finite cardinality $|K_n| = p^n$.
- $K_n \subset K_m$ if n divides m.
- $K = \bigcup_{n \geq 1} K_n$.

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It follows that K is the increasing union of the finite subfields $L_n := K_{n!}$, $n \geq 1$.

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It follows that K is the increasing union of the finite subfields $L_n := K_{n!}$, $n \geq 1$.
Let $A, B \in \mathrm{Mat}_d(K[G])$ such that $AB = I_d$.
There exists $n_0 \geq 1$ such that $A, B \in \mathrm{Mat}_d(L_{n_0}[G])$.
Then $BA = I_d$ by Case 1.

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Case 3: the field K is algebraically closed with characteristic $p > 0$

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Case 4: the field K is algebraically closed with characteristic 0

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Case 3: the field K is algebraically closed with characteristic $p > 0$ Apply Lemma 2, Case 2, and the first Lefschetz principle.

Case 4: the field K is algebraically closed with characteristic 0 Suppose by contradiction that $K[G]$ is not stably finite. Then apply Lemma 1, the second Lefchetz principle, and Case 3.

Proof of Theorem A (continued)

Case 3: the field K is algebraically closed with characteristic $p > 0$ Apply Lemma 2, Case 2, and the first Lefschetz principle.

Case 4: the field K is algebraically closed with characteristic 0 Suppose by contradiction that $K[G]$ is not stably finite. Then apply Lemma 1, the second Lefchetz principle, and Case 3.

Case 5: K is an arbitrary field

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Case 4: the field K is algebraically closed with characteristic 0 Suppose by contradiction that $K[G]$ is not stably finite. Then apply Lemma 1, the second Lefchetz principle, and Case 3.

Case 5: K is an arbitrary field Consider the algebraic closure L of K.
The group algebra $L[G]$ is stably finite by Case 3 and Case 4.
As $K[G] \subset L[G]$, we deduce that $K[G]$ is itself stably finite. □