First-order model theory, surjunctivity, and Kaplansky's stable finiteness conjecture

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This is joint work:

T.Ceccherini-Silberstein, M. Coornaert, and X.K. Phung: "First-order model theory and Kaplansky's stable finiteness conjecture for surjunctive groups", arXiv:2310.09451, to appear in *Groups, Geometry, and Dynamics*.

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Definition

A ring R is called directly finite if

 $\forall a, b \in R, ab = 1 \implies ba = 1.$

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For a ring R, the following conditions are equivalent:

- *R* is directly finite;
- every left-invertible element in R is invertible;
- every right-invertible element in R is invertible;
- *R* is Hopfian as a left *R*-module;
- *R* is Hopfian as a right-module.

(A module M is called *Hopfian* if every surjective endomorphism of M is an automorphism.)

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Stably finite rings

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A ring R is called stably finite if the ring $Mat_d(R)$ (ring of $d \times d$ matrices with entries in R) is directly finite for any $d \ge 1$.

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For a ring R, the following conditions are equivalent:

- *R* is stably finite;
- $\ \ \, \textcircled{} \quad \forall d\geq 1, \forall A,B\in \mathsf{Mat}_d(R), \quad AB=I_d \implies BA=I_d,$
- $\forall d \ge 1$, the left *R*-module R^d is Hopfian;
- $\forall d \geq 1$, the right *R*-module R^d is Hopfian;
- every finitely generated free left *R*-module is Hopfian;
- every finitely generated free right *R*-module is Hopfian.

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Examples of stably finite rings

- Any finite ring is stably finite.
- Any commutative ring is stably finite.
- Any field is stably finite.
- Any division ring is stably finite.
- Any left (or right) Noetherian ring is stably finite.
- If V is a vector space over a field K then End_K(V) is stably finite iff dim_K(V) < ∞.
- Any unit-regular ring is stably finite.

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Stable finiteness vs direct finiteness

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R stably finite \implies *R* directly finite (since Mat₁(*R*) = *R*). There exist directly finite rings that are not stably finite. For any $d \ge 1$, there exist rings *R* such that Mat_d(*R*) is directly finite but Mat_{d+1}(*R*) is not.

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Group algebras

Let G be a group and let K be a field.

The group algebra of G with coefficients in K is the K-algebra K[G] constructed as follows:

- *K*[*G*] is the *K*-vector space with base *G*;
- the multiplication on K[G] is obtained by extending K-linearly the group operation on G.

Thus, every $\alpha \in K[G]$ can be uniquely written in the form

$$\alpha = \sum_{g \in G} \alpha_{gg}$$

with $\alpha_g \in K$ for all $g \in G$ and $\alpha_g = 0$ for all but finitely many $g \in G$.

The operations on K[G] are given by the formulae:

$$\begin{aligned} \alpha + \beta &= \sum_{g \in G} (\alpha_g + \beta_g) g, \\ \lambda \alpha &= \sum_{g \in G} (\lambda \alpha_g) g, \\ \alpha \beta &= \sum_{g \in G} \left(\sum_{h_1, h_2 \in G: h_1 h_2 = g} \alpha_{h_1} \beta_{h_2} \right) g \end{aligned}$$

for all $\alpha, \beta \in K[G]$ and $\lambda \in K$.

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Kaplansky's stable finiteness conjecture

Theorem (Kaplansky)

Let G be a group and let K be a field of characteristic 0. Then the group algebra K[G] is stably finite.

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Theorem (Kaplansky)

Let G be a group and let K be a field of characteristic 0. Then the group algebra K[G] is stably finite.

The proof is analytical after reducing to the case $K = \mathbb{C}$. Kaplansky's stable finiteness conjecture: The group algebra K[G] is stably finite for every group G and every field K.

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Let G be a group and let A be a finite set.

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$$A^G \coloneqq \{x \colon G \to A\}$$

is equipped with the G-shift and the prodiscrete topology defined as follows.

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where $L_{g^{-1}}: G \to G$ is the left multiplication by g^{-1} . The prodiscrete topology on A^G is the product topology obtained by taking the discrete topology on every factor A of $A^G = \prod_{g \in G} A$. The *G*-shift on A^G is continuous. The space A^G is homeomorphic to the Cantor space for $|A| \ge 2$ and G countably infinite.

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The notion of a surjunctive group goes back to Gottschalk.

Definition

A group G is called surjunctive if, for any finite set A and every continuous G-equivariant map $\tau: A^G \to A^G$, one has

 τ injective $\implies \tau$ surjective.

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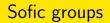
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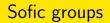
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No example of a non-surjunctive group has been found up to now. Gottschalk's conjecture: Every group is surjunctive.



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Roughly speaking, a group is sofic if it can be "well approximated" by finite symmetric groups.

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All finite groups, all residually finite groups, all abelian groups, all nilpotent groups, all solvable groups, all amenable groups, all residually amenable groups, all linear groups are sofic.

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Theorem (Gromov and Weiss)

Every sofic group is surjunctive.

Given a non-empty finite set X, let Sym(X) denote the symmetric group of X.

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Definition

A group G is called sofic if for every $\varepsilon > 0$ and for every finite subset $F \subset G$, there exist a non-empty finite set X and a map $\phi \colon F \to \text{Sym}(X)$ such that

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Stable finiteness of group algebras of surjunctive groups

The following result was obtained by Xuan Kien Phung using algebraic geometry.

Theorem A (Phung)

Every surjunctive group satisfies Kaplansky's stable finiteness conjecture.

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As sofic \implies surjunctive by the Gromov-Weiss theorem, we get.

Corollary (Elek et Szabó)

Every sofic group satisfies Kaplansky's stable finiteness conjecture.

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Two fields are called elementary equivalent if they satisfy the same first-order sentences in the language of rings $L = \{+, -, \times, 0, 1\}$.

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Two fields are called elementary equivalent if they satisfy the same first-order sentences in the language of rings $L = \{+, -, \times, 0, 1\}$.

Examples

• The sentence $\forall x, \exists y, x = y^3$ is satisfied by \mathbb{R} but not by \mathbb{Q} . Thus, the fields \mathbb{R} and \mathbb{Q} are not elementary equivalent.

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Theorem (Second Lefschetz principle)

Let ψ be a first-order sentence in the language of rings which is satisfied by some (and hence any) algebraically closed field of characteristic 0. Then there exists an integer N such that ψ is satisfied by every algebraically closed field of characteristic $p \ge N$.

T.Ceccherini-Silberstein (Rome)

Proof of Theorem A

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Lemma 1

Let G be a group, let $d \ge 1$ be an integer, and let S be a finite subset of G. Then there exists a first-order sentence ψ in the language of rings such that a field K satisfies ψ if and only if there exist matrices $A, B \in Mat_d(K[G])$ such that

- **(**) the support of each entry of A and of each entry of B is contained in S;

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- the support of each entry of A and of each entry of B is contained in S;

Lemma 2

Let G be a group and suppose that K and L are elementary equivalent fields. Then K[G] is stably finite if and only if L[G] is stably finite.

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Proof of Theorem A (continued)

T.Ceccherini-Silberstein (Rome) Model theory, surjunctivity, and Kaplansky conjecture

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Let G be a surjunctive group and let K be a field.

Case 1: the field K is finite

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Proof of Theorem A (continued)

T.Ceccherini-Silberstein (Rome) Model theory, surjunctivity, and Kaplansky conjecture

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Case 2: the field K is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime p

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Case 2: the field K is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime p Consider the Frobenius automorphism $\phi \colon K \to K$ defined by

$$orall \lambda \in \mathcal{K}, \quad \phi(\lambda) \coloneqq \lambda^{
ho}$$

For $n \geq 1$, define $K_n \subset K$ by

$$K_n := \operatorname{Fix}(\underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}).$$

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•
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.

It follows that K is the increasing union of the finite subfields $L_n := K_{n!}, n \ge 1$. Let $A, B \in Mat_d(K[G])$ such that $AB = I_d$. There exists $n_0 \ge 1$ such that $A, B \in Mat_d(L_{n_0}[G])$. Then $BA = I_d$ by Case 1.

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Case 3: the field K is algebraically closed with characteristic p > 0

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Case 4: the field K is algebraically closed with characteristic 0

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Case 5: K is an arbitrary field

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Case 4: the field K is algebraically closed with characteristic 0 Suppose by contradiction that K[G] is not stably finite. Then apply Lemma 1, the second Lefchetz principle, and Case 3.

Case 5: K is an arbitrary field Consider the algebraic closure L of K. The group algebra L[G] is stably finite by Case 3 and Case 4. As $K[G] \subset L[G]$, we deduce that K[G] is itself stably finite.

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