Graphs and Groups, Complexity and Convexity (G2C2-2024)

(Summer School, August 11-25, 2024, Hebei Normal University, Shijiazhuang, China)

Title: Combinatorial convexity

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Description: This course is about combinatorial properties of convex sets, of families of convex sets in d-dimensional Euclidean spaces, and properties of finite points sets related to convexity. Typical examples are the classical theorems of Helly, Carathéodory, and Radon. More recent results of the same type are Tverberg's theorem, the fractional Helly theorem, the colourful Carathéodory and Helly theorems, the weak epsilon-net result, and the (p,q) theorem of Alon and Kleitman.

Outline of the course:

Lecture 1. Basic concepts.
We start with definitions and basic concepts in convexity. We then state and prove Carathéodory's theorem (from 1904) in various forms and its extensions.

Lecture 2. Radon's theorem.
The next topic is Radon's theorem. We prove its original (linear) version and also its topological form. Several applications of Radon's theorem will also be discussed.

Lecture 3. Helly's theorem.
One of the most important theorems in combinatorial convexity is Helly's theorem (1914). It has several proofs and many applications. We will explain some of them.

Lecture 4. Colourful Carathéodory and Helly.
Colourful versions of the theorems of Carathéodory and Helly are important and useful. We give their proofs and explain several of their applications.

Lecture 5. Tverberg's theorem.
Tverberg's theorem, a gem in discrete geometry is from 1966 and is a central result in combinatorial convexity. It generalizes Radon's theorem. I will state and prove this beautiful theorem and explain how it can be applied.

Lecture 6. The covering number.
Carathéodory's theorem says that the convex hull of a finite set X in R^d is covered by the simplices of X, that is, by sets of the form conv Y where Y is a subset of X of size at most d+1. How many times is a point a in the convex hull of X covered? Which is the maximally covered point and how many times is it covered?

Lecture 7. Weak epsilon nets.
In combinatorics epsilon nets are important. Their convex version is the weak epsilon net. A weak epsilon net for a given set X of n points in R^d, and $\varepsilon>0$, is a set S in R^d that intersects the convex hull of every subset Y of X of size at least $\varepsilon|X|$. The question is how small size a weak epsilon net can have.

Lecture 8. The (p,q) theorem.
This result is a far reaching generalization of Helly's theorem. A finite family F of convex sets in Rd has the (p,q) property if among every p sets of the family there are q that have a point in common. We assume here that $p \geq q \geq d+1$, so the case $p=q=d+1$ is just the condition of the Helly theorem. The (p,q) condition guarantees the existence of a finite set (whose size only depends on p, q, and d) that intersects every set in the family.

Bibliography

Imre Bárány, Combinatorial Convexity, American Mathematical Society (2022).