G2C2 Lecture No. 8. Multivariate P-and Q-polynomial association schemes

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Basic References

[1] Bernard-Crampé-d'Andecy-Vinet-Zaimi: Bivariate P-polynomial association schemes, Algebraic Combinatorics (2024).

[2] Bannai-Kurihara-Zhao-Zhu: Multivariate P-and/or Q-polynomial association schemes, arXiv:2305.00708v2.

[3] Ceccerini-Silberstein, F. Scarabotti and F. Tolli: Trees, wreath products and finite Gelfand pairs, Advances in Math. (2006)

The content of this part of my talk is based on the joint work with
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§1. Introduction.

The classification problem of P-and Q-polynomial association schemes has been one of the most important problems in algebraic combinatorics. Cf. Bannai-Ito, Algebraic Combinatorics I : Association Schemes, Benjamin-Cummings (1984).

The theorem of Leonard (1982) that the spherical functions (and the character table) of P-and Q-polynomial association schemes can be described by Askey-Wilson orthogonal polynomials and their relatives (namely including special cases and limiting cases) was the important starting point.

These are one variable (discrete) orthogonal polynomials. So, the following questions naturally arise.

(i) Is there a good generalization of the concept of P-and Q-polynomial association schemes? (Namely, is there a concept of higher rank P-and Q-polynomial association schemes?)

(ii) What are the orthogonal polynomials appearing in the higher rank P-and Q-polynomial association schemes? (Namely, is there a higher rank analogue of the Leonard's theorem?)

Frankly speaking, for (i) there have been only very limited studies on higher rank P- and Q-polynomial association schemes at the level of association schemes. This was because the concept was not well formulated, and not much examples were explicitly known, except for in some work of Mizukawa-Tanaka, Gasper-Rahman, Iliev-Terwillger, etc. (Mostly, symmetrizations of association schemes.) While, for (ii) at the level of orthogonal polynomials, multivariate version of Askey-Wilson polynmials have been known, see e.g., Tratnik. However, these are very special cases of generalizations, and they do not cover general cases of multivariate orthogonal polynomials corresponding to higher rank P-and Q-polynomial association schemes. Namely, higher rank version of the theorem of Leonard is still very far away, in my opinion. (I will specify this remark later in my talk.)

In this talk, we first discuss the concept of multivariate P-and Q- polynomial association schemes, then discuss some families of explicit examples, and finally we mention many speculations on which directions this research should proceed.

The first very successful attempt was made by the recent work of [BCdVZ] Bernard, Crampé, d'Andecy, Vinet, Zaimi: Bivariate P-polynomial association schemes, arXiv:2212.10824 (just published in Algebraic Combinatorics, 2024). They [BCdVZ] defined the concept of bivariate P-polynomial association schemes of type (α, β) as well as Q-polynomial association schemes of type (α, β), and studied these concept as well as many such examples.

What I want to talk today is our attempts to generalize their work. Our research was strongly motivated by their work [BCdVZ].
In particular,
(i) We want to define a similar concept in a more general context, i.e., for any monomial order beyond (α, β)-type.
(ii) We want to consider general multivariate case beyond the bivariate case.

Our paper [BKZZ] Bannai, Kurihara, Zhao, Zhu: Multivariate P- and/or Q-polynomial association schemes, is seen in arXiv: 2205.00707v2.

§2. Multivariate P- or Q-polynomial association schemes

Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of natural numbers, and let

$$\mathbb{N}^\ell := \{(n_1, n_2, ..., n_\ell) \mid n_i \in \mathbb{N}\}.$$

Definition 2.1. (monomial order). A monomial order $\leq$ is an order on $\mathbb{N}^\ell$ satisfying the following three conditions (a), (b) and (c).
(a) $\leq$ is a total order,
(b) For $\alpha, \beta, \gamma \in \mathbb{N}^\ell$, if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.
(c) $\leq$ is a well ordering, i.e., any non-empty subset of $\mathbb{N}^\ell$ has a minimum element under $\leq$.

Examples of monomial orders.
• lex order (lexicographic order). Let $\alpha = (n_1, n_2, ..., n_\ell)$, $\beta = (m_1, m_2, ..., m_\ell) \in \mathbb{N}^\ell$. We define $\alpha \leq_{lex} \beta$ if the left most nonzero entry of $\alpha - \beta \in \mathbb{Z}^\ell$ is negative.
• grlex order (graded lexicographic order). $\alpha \leq_{grlex} \beta$, if $|\alpha| < |\beta|$ or both $|\alpha| = |\beta|$ and $\alpha \leq_{lex} \beta$ hold, where $|\alpha| = n_1 + n_2 + \cdots + n_\ell$ and $|\beta| = m_1 + m_2 + \cdots + m_\ell$.

The concept of bivariate P-polynomial association scheme (as well as Q-polynomial association scheme) was defined by BCdVZ with respect to the so called (α, β)-order, where $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$. Motivated by their work, we have succeeded in defining a similar concept for any general monomial order. This new definition does not cover all the case of (α, β)-order by BCdVZ completely, but more general. We believe that this new definition is conceptually very natural and has the advantage of working for arbitrary ℓ (not just for the bivariate case).

For $\alpha = (n_1, n_2, ..., n_\ell) \in \mathbb{N}^\ell$ and $x = (x_1, x_2, ..., x_\ell)$, we write the monomial $x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}$ by x^α. Then α is called the multidegree of x^α.

Definition 2.2. (The original definition in our arXiv paper). Let $\mathcal{D} \subset \mathbb{N}^{\ell}$, $e_i = (0, ..., 0, 1, 0, ..., 0) \in \mathcal{D}$ (the ith coordinate is 1), and let $\leq$ be a monomial order in $\mathbb{N}^{\ell}$. A commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{(i=0,1,...,d)})$ is called ℓ-variate P-polynomial on the domain $\mathcal{D}$ with respect to the monomial order $\leq$, if the following three conditions (i),(ii) and (iii) are satisfied.

Condition (i). If $(n_1, n_2, ..., n_\ell) \in \mathcal{D}$ and $0 \leq m_i \leq n_i (i = 1, 2, ..., \ell)$. Then $(m_1, m_2, ..., m_\ell) \in \mathcal{D}$.

Condition (ii). There exists a relabelling of the adjacency matrices of $\mathfrak{X}$: $\{A_0, A_1, ..., A_d\} = \{A_\alpha\}_{\alpha \in \mathcal{D}}$ such that for each $\alpha \in \mathcal{D}$, $A_\alpha = v_\alpha(A_{e_1}, A_{e_2}, ..., A_{e_\ell})$ where the ℓ-variate P-polynomial $v_\alpha(x)$ is expressed as $v_\alpha(x) = \sum_{\beta \in \mathcal{D}} c_\beta x^\beta$ with $\beta \leq \alpha$ and $c_\alpha \neq 0$.

Condition (iii). For $i = 1, 2, ..., \ell$ and $\alpha = (n_1, n_2, ..., n_\ell) \in \mathcal{D}$, the product $A_{e_i}\ A_{e_1}^{n_1} A_{e_2}^{n_2} \cdots A_{e_\ell}^{n_\ell}$ is a linear combination of

$$\{A_{e_1}^{m_1} A_{e_2}^{m_2} \cdots A_{e_\ell}^{m_\ell} \mid \beta = (m_1, m_2, ..., m_\ell) \in \mathcal{D}, \beta \leq \alpha + e_i\}.$$

Motivated by the work of BCdVZ, we get the following:

Proposition 2.3. Let $\mathcal{D} \subset \mathbb{N}^\ell$, and let $e_1, e_2, \ldots, e_\ell$ be in $\mathcal{D}$. Let $\mathfrak{X} = (X, \{A_\alpha\}_{\alpha \in \mathcal{D}})$ be a commutative association scheme. Then the following two statements are equivalent.
(i) $\mathfrak{X}$ is an ℓ-variate P-polynomial association scheme (with the relations indexed by $\mathcal{D}$) with respect to a monomial order $\leq$.
(ii) The condition (i) of Definition 2.2 holds for $\mathcal{D}$ and the intersection numbers satisfy for each $i = 1, 2, \ldots, \ell$ and each $\alpha \in \mathcal{D}$, $p^{\beta}_{e_i,\alpha} \neq 0$ for $\beta \in \mathcal{D}$ implies $\beta \leq \alpha + e_i$. Moreover, if $\alpha + e_i \in \mathcal{D}$, then $p^{\alpha+e_i}_{e_i,\alpha} \neq 0$ holds.

Similarly, we can define multivariate Q-polynomial association schemes.

Definition 2.4. Let $\mathcal{D}^* \subset \mathbb{N}^\ell$, $e_i = (0, ..., 0, 1, 0, ..., 0) \in \mathcal{D}^*$ (the ith coordinate is 1), and let $\leq$ be a monomial order in $\mathbb{N}^\ell$. A commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{(i=0,1,...,d)})$ is called ℓ-variate Q-polynomial on the domain $\mathcal{D}^*$ with respect to the monomial order $\leq$, if the following three conditions (i),(ii) and (iii) are satisfied.

Condition (i). If $(n_1, n_2, ..., n_\ell) \in \mathcal{D}^*$ and $0 \leq m_i \leq n_i (i = 1, 2, ..., \ell)$. Then $(m_1, m_2, ..., m_\ell) \in \mathcal{D}^*$.

Condition (ii). There exists a relabelling of the primitive idempotents of $\mathfrak{X}$: $\{E_0, E_1, ..., E_d\} = \{E_\alpha\}_{\alpha \in \mathcal{D}^*}$ such that for each $\alpha \in \mathcal{D}^*$, $|X|E_\alpha = v^*_\alpha(|X|E_{e_1}, |X|E_{e_2}, ..., |X|E_{e_\ell})$ where $v^*_\alpha(x)$ is expressed as $v^*_\alpha(x) = \sum_{\beta \in \mathcal{D}^*} c^*_\beta x^\beta$ with $\beta \leq \alpha$ and $c^*_\alpha \neq 0$. (Here, the multiplication is Hadamard product $\circ$.)

Condition (iii). For $i = 1, 2, ..., \ell$ and $\alpha = (n_1, n_2, ..., n_\ell) \in \mathcal{D}^*$, the product $(E_{e_i}) \circ (E_{e_1})^{n_1} \circ (E_{e_2})^{n_2} \circ \cdots \circ (E_{e_\ell})^{n_\ell}$ is a linear combination of

$$\{E_{e_1}^{m_1} \circ E_{e_2}^{m_2} \circ \cdots \circ E_{e_\ell}^{m_\ell} \mid \beta = (m_1, m_2, ..., m_\ell) \in \mathcal{D}^*, \beta \leq \alpha + e_i\}.$$

Proposition 2.5. Let $\mathcal{D}^* \subset \mathbb{N}^\ell$, and let $e_1, e_2, ..., e_\ell$ be in $\mathcal{D}^*$. Let $\mathfrak{X}$ be a commutative association scheme. Then the following two statements are equivalent.
(i) $\mathfrak{X}$ is an ℓ-variate Q-polynomial association scheme (with the primitive idempotents indexed by $\mathcal{D}^*$) with respect to a monomial order $\leq$.
(ii) The condition (i) of Definition 2.4 holds for $\mathcal{D}^*$ and the dual intersection numbers satisfy for each $i = 1, 2, ..., \ell$ and each $\alpha \in \mathcal{D}^*$, $q^{\beta}_{e_i,\alpha} \neq 0$ for $\beta \in \mathcal{D}^*$ implies $\beta \leq \alpha + e_i$. Moreover, if $\alpha + e_i \in \mathcal{D}^*$, then $q^{\alpha+e_i}_{e_i,\alpha} \neq 0$ holds.

Discussions of some explicit examples.

We consider the following families of examples, mostly following T. Ceccerini-Silberstein, F. Scarabotti and F. Tolli: Trees, wreath products and finite Gelfand pairs, Advances in Math. (2006).

I. Direct products.

II. Compositions.

III. A.S. coming from attenuated spaces.

IV. A.S. coming from isotropic subspaces.

V. Symmetrizations. (Extensions, or Generalized Hamming schemes.)

VI. Generalized Johnson schemes.
(Including non-binary Johnson schemes.)

We study each family more closely. (They are not exhaustive.)

I. Direct products.

The direct product of two P- (or Q-)polynomial association schemes becomes obviously a bivariate P- (or Q-)polynomial association scheme, in the sense of BCdVZ and also in the sense of BKZZ.

II. Composition of Gelfand pairs.

Let (G, K) and (F, H) be Gelfand pairs of finite groups. Let $X = G/K$ and $Y = F/H$. Let $F \wr G$ be the wreath product of F by G. Namely,

$$F^X \times G = \{(f, g) \mid f : X \to F, g \in G\},$$

and $(f, g)(f', g') = (f \cdot (gf'), gg')$, where $[f \cdot (gf')](x) = f(x)f'(g^{-1}x)$ for all $x \in X$. We consider the action of the wreath product group $F \wr G$ on $X \times Y$ by $(f, g)(x, y) = (gx, f(gx)y)$, for $(f, g) \in F \wr G$ and $(x, y) \in X \times Y$. Let $x_0 \in X$ be the fixed point in X by K, and let $y_0 \in Y$ be the fixed point by F. Then the stabilizer $J \leq G$ of the point (x_0, y_0) by the action of $F \wr G$ on $X \times Y$ is given by

$$J = \{(f, k) \in F \wr G \mid k \in K, f(x_0) \in H\}.$$

Let $X = \cup_{i=0}^{n} \Xi_i$ and $Y = \cup_{j=0}^{m} \Gamma_j$ be the decomposition of X and Y by the actions of G and F respectively (with $\Xi_0 = \{x_0\}$ and $\Gamma_0 = \{y_0\}$). Then the decomposition of $X \times Y$ by J-orbits is given by

$$X \times Y = [\cup_{j=0}^{m} (\Xi_0 \times \Gamma_j] \cup [\cup_{i=1}^{n} (\Xi_i \times Y)].$$

Suppose that (G, K) and (F, H) are Gelfand pairs and let $L(X) = \oplus_{i=0}^{n} V_i$ and $L(Y) = \oplus_{j=0}^{m} W_j$ be the decomposition into G-(respectively F-) irreducible representations, where V_0 and W_0 are the one-dimensional subspaces of constant functions. It is known that $(F \wr G, J)$ is a Gelfand pair. Moreover, the decomposition of $L(X \times Y)$ into $(F \wr G)$-irreducible subspaces is given by

$$L(X \times Y) = [\oplus_{i=0}^{n} (V_i \otimes W_0)] \oplus [\oplus_{j=1}^{m} (L(X) \otimes W_j)].$$

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the association schemes obtained by the Gelfand pairs (G, K) and (F, H), respectively. Then the association scheme obtained by the Gelfand pair $(F \wr G, J)$ is called the composition of $\mathfrak{X}$ and $\mathfrak{Y}$. Note that the composition of $\mathfrak{X}$ and $\mathfrak{Y}$ is a fusion scheme of the direct product association scheme $\mathfrak{X} \otimes \mathfrak{Y}$.

<u>Theorem 3.1.</u> Let $\mathfrak{Z}$ be the composition of $\mathfrak{X}$ and $\mathfrak{Y}$.
(i) If $\mathfrak{X}$ and $\mathfrak{Y}$ are P-polynomial association schemes, then $\mathfrak{Z}$ is a bivariate P-polynomial association scheme on

$$\mathcal{D} = \{(i,0)\}_{i=1}^{n} \cup \{(0,j)\}_{j=0}^{m} \subset \mathbb{N}^2$$

worth respect to $\leq_{lex}$.
(ii) If $\mathfrak{X}$ and $\mathfrak{Y}$ are Q-polynomial association schemes, then $\mathfrak{Z}$ is a bivariate Q-polynomial association scheme on

$$\mathcal{D}^* = \{(j,0)\}_{j=1}^{m} \cup \{(0,i)\}_{i=0}^{n} \subset \mathbb{N}^2$$

with respect to $\leq_{lex}$.

III. A.S. coming from attenuated spaces.

Let $\mathbb{F}_q^{D+L}$ be the $(D+L)$-dim vector space over the finite field $\mathbb{F}_q$. Let W be a fixed subspace of dimensin L. Let X be the set of all d-dimensional subspaces V with $V \cap W = \{0\}$. (So, obviously $d \leq D$.)
Let $\mathcal{D}$ be the index set defined by

$$\mathcal{D} = \{(i, j) \in \mathbb{N}^2 \mid i \leq min(d, D-d), j \leq min(d-i, L)\}.$$

(This domain is a triangle shape if $d \leq L$ and $D \geq 2d$.) For $U, U' \in X$, let $R_{i,j}$ be defined by $(U, U') \in R_{i,j}$ if and only if $\dim((U+W)/W \cap (U'+W)/W) = m - i$ and $\dim(U \cap U') = (d-i) - j$. Then $\mathfrak{X} = (X, \{R_{i,j}\})$ with $(i, j) \in \mathcal{D}$ becomes a symmetric association scheme, called the association scheme coming from attenuated space.

BCdVZ proved that, if $d \leq L$, it is bivariate P-polynomial of type $(1, 0)$. We point out that this association scheme is also shown to be a bivariate P-polynomial in our sense with respect to the order $\leq_{grlex}$ even if $L < d$.

Actually, we get that $\mathcal{D}^* = \mathcal{D}$ in this case, where $\mathcal{D}^*$ is the set parametrizing the primitive idempotents of the association scheme.

We want to explain the following result.

Theorem 3.2. [BKZZ2]. The association scheme coming from attenuated space is bivariate Q-polynomial association scheme in our sense (with respect to the $\leq_{grlex}$) for any pair of L and d, i.e., even if $L < d$). Therefore, the association scheme coming from attenuated space is always bivariate P- and Q-polynomial association scheme (with respect to the $\leq_{grlex}$).

[BKZZ2] Bannai-Kurihara-Zhao-Zhu, Bivariate Q-polynomial structures for the nonbinary Johnson scheme and the association scheme obtained from attenuated spaces, arXiv:2403.05169 (just accepted in J. of Algebra).

IV. Association schemes coming from isotropic subspaces.

Consider the vector space $\mathbb{F}_q^D$ with a non-degenerate form, i.e., alternating form, symmetric form, hermitian form, etc.. Let X be the set of isotropic subspaces of dimension d. For $V, V' \in X$, let

$$R_{i,j} = \{(V, V') \in X \times X \mid dim(V \cap V') = d - i - j, dim(V^{\perp} \cap V') = d - i\}.$$

$$\mathcal{D} = \{(i, j) \mid 0 \leq i \leq d - j, 0 \leq j \leq min(d, D - d)\}.$$

Then we have a commutative (symmetric) association scheme with the parameter set $\mathcal{D}$. BCdVZ proved that it is generally bivariate P-polynomial association scheme in their sense. (It is also shown that it is generally bivariate P-polynomial association scheme in our sense.

The spherical functions are explicitly calculated by Stanton (1980) for most cases explicitly, but extremely involved. It is conjectured that they are also bivariate Q-polynomial, but it is not yet proved in general. (A very important open problem.)

V. Symmetrizations. (Extensions, or Generalized Hamming schemes.)

Le $\mathfrak{X} = (X, \{R_i\}_{i=0}^{\ell})$ be a commutative association scheme of class ℓ. Let n be a positive integer. For $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X^n$. For each $i (0 \leq i \leq \ell)$ set $\tau_i(x, y) = |\{t = 1, 2, ..., n \mid (x_t, y_t) \in R_i\}|$. Then we define the ℓ-tuple

$$\mathcal{R}(x, y) = (\tau_1(x, y), \tau_2(x, y), ..., \tau_\ell(x, y)).$$

Then all $\mathcal{R}(x, y)$ are in $D = \{\alpha \in \mathbb{N}^{\ell} \mid |\alpha| \leq n\}$. Then $\mathcal{S}^n(\mathfrak{X}) = (X^n, \mathcal{R})$ is a commutative association scheme and called extension (or symmetrization) of $\mathfrak{X}$ of length n. $\mathcal{S}^n(\mathfrak{X})$ is a fusion scheme of the n-times direct product $\bigotimes \mathfrak{X}^n$ of $\mathfrak{X}$.

<u>Theorem 3.3</u> $\mathcal{S}^n(\mathfrak{X})$ is an ℓ-variate P-polynomial and Q-polynomial association scheme on $\mathcal{D}$ with respect to $\leq_{grlex}$.

Note that if the association scheme $\mathfrak{X}$ is a Gelfand pair (F, H), then $\mathcal{S}^n(\mathfrak{X})$ is a Gelfand pair $(F \wr S_n, H \wr S_n)$.

We remark that the spherical functions (as well as character tables) of such association schemes are calculated, i.e., expressed by certain hypergeometric series, Aomoto-Gelfand hypergeometric functions, see e.g., Mizukawa-Tanaka, etc.

VI. Generalized Johnson schemes.
(Including non-binary Johnson schemes.)

First, we consider the special case of non-binary Johnson association scheme, $J_r(n,k)$.

Let n and r be positive integers with $r \geq 2$ and let k be a natural number such that $0 \leq k \leq n$. Let $K = \{0, 1, ..., r-1\}$ be a set of cardinality r. For a vector $x = (x_1, x_2, ..., x_n) \in K^n$, its weight $w(x)$ is defined by the number of nonzero entries. Let $S = \{x \in K^n \mid w(x) = k\}$. Then $|S| = (r-1)^k \binom{n}{k}$. Let $\mathcal{D} = \{(i,j) \mid i+j \leq k, 0 \leq i \leq min(k, n-k)\}$. Let $R_{i,j}$ be defined as $(x,y) \in R_{i,j}$, if $|\{i \mid x_i = y_i \neq 0\}| = k-i-j$ and $|\{i \mid x_i \neq 0, y_i \neq 0\}| = k-j$. Then, $(S, \{R_{i,j}\})$ with $(i,j) \in \mathcal{D}$ becomes a commutative (symmetric) association scheme called non-binary Johnson association schemes $J_r(n,k)$. (If $r = 2$, then $J_r(n,k)$ is the Johnson scheme $J(n,k)$ and if $k = n$, then $J_r(n,k)$ is the Hamming association scheme $H(n,r)$.)

It was shown by BCdVZ that the non-binary Johnson association scheme is bivariate P-polynomial association scheme (of type $(1, 0)$). Crampé-Vinet-Zaimi-Zhang: A bivariate Q-polynomial structure for the nonbinary Johnson scheme (JCT(A), 2024) proved that non-binary Johnson association scheme is bivariate Q-polynomial, if $2k \leq n$, namely, the domain $\mathcal{D}$ is of triangular type. Then, [BKZZ2]: Bivariate Q-polynomial structures for the nonbinary Johnson scheme and the association scheme obtained from attenuated spaces, arXiv:2403.05169 (just accepted in J. of Algebra) proved that non-binary Johnson association scheme is always bivariate Q-polynomial (for any shape of $\mathcal{D}$), i.e., even if $2k > n$.

Now, we want to consider generalized Johnson schemes in general.

Let (F, H) be a Gelfand pair, and let $Y = F/H$ with $y_0 \in Y$ being the point stabilized by H. Let $Y = \cup_{i=0}^{m} \Lambda_i$ be the decomposition of Y into its H-orbits with $\Lambda_0 = \{y_0\}$. Let $0 \leq h \leq n$. Let $\Omega_h (= S_n/(S_{n-h} \times S_h) \cong J(n, k))$ be the set of h-elements subset of $\{1, 2, ..., n\}$. Let $\Theta_h = Y^{\Omega_h} = \cup_{A \in \Omega_h} Y^A =$ the set of all functions θ from Ω_h to Y.

Then $F \wr S_n$ acts naturally on Θ_h transitively, with the stabilizer of the point is $\cong (H \wr S_h) \times (F \wr S_{n-h})$.

(Exactly speaking, if $(f, \pi) \in F \wr S_n$ and $\theta \in \Theta_h$ then $(f, \pi)(\theta)$ is the function, with domain π $domain(\theta)$ defined by

$$[(f, \pi)\theta](j) = f(j)\theta(\pi^{-1} j).$$

for every $j \in \pi$ $domain(\theta)$.)

Now, the relations $\mathcal{R}$ of this Gelfand pair Θ_h are parametrized by the set

$$\{(t, a_0, a_1, ..., a_m) \mid 0 \leq t \leq \min\{h, n-h\} \text{ with } a_0 + a_1 + \cdots + a_m = h - t\}.$$

Thus, $\mathfrak{X} = (\Theta_h, \mathcal{R})$ becomes a commutative association scheme parametrized by

$$\mathcal{D} = \{(t, a_1, ..., a_m)\mathbb{N}^{m+1} \mid 0 \leq t \leq \min\{h, n-h\}, a_i \geq 0, \text{ with } a_0 + a_1 + \cdots + a_m = h - t\}.$$

Exactly speaking, for θ_1 and $\theta_2 \in \Theta_h$, we have $|domain(\theta_1) \cap domain(\theta_2)| = h - t$ and

$$a_i = |\{a \in \ domain(\theta_1) \cap \ domain(\theta_2) \mid (\theta_1(a), \theta_2(a)) \in \overline{\Lambda_i}\}|$$

for $i = 0, 1, ..., m$.

<u>Theorem 3.4</u> (BKZZ). If (F, H) is a Gelfand pair of class ℓ, then $(F \wr S_n, (H \wr S_h) \times (F \wr S_{n-h}))$ becomes a $(\ell + 1)$-variate P-polynomial association scheme in our sense.

Note that (F, H) need not to be P-polynomial, i.e., just any Gelfand pair. (This is a bit surprizing!) We conjecture that a similar result will hold for Q-polynomial case, but it is not yet proved. (An important open problem!) We mention that the non-binary Johnson scheme $J(v, n)$ is the special case of generalized Johnson association scheme with $(F, H) = (S_{q-1}, S_{q-2})$.

Speculations

Multivariate P and Q-polynomial association scheme from root system

Iliev-Terwilliger: The Rahman polynomials and the Lie algebra $sl_3(\mathbb{C})$, Trans. AMS (2012), considers some multivariate P-polynomial (and/or Q-polynomial) association schemes from the root system, in particular of type A_n and possibly for other types. Here, we give the definition of A_M-Leonard pair, following Iliev-Terwilliger and Crampé-Zaimi: Factorized A_M-Leonard pair (arXiv:2312.08312).

Let $\mathbb{F}$ denote a field, and let V be a finite dimensional vector space over $\mathbb{F}$. Let $End(V)$ be the set consisting of $\mathbb{F}$-linear maps from V to V. For integers $M, N \geq 1$, let

$$\mathcal{D} = \{\alpha \in \mathbb{N}^M \mid |\alpha| \leq N\}.$$

(So, $\mathcal{D}$ is an isosceles right triangle shape and $|\mathcal{D}| = \binom{M+N}{M}$.)

A pair of elements $(r_1, ..., r_M)$ and $(r'_1, ..., r'_M)$ in $\mathcal{D}$ is called adjacent if $(r_1 - r'_1, ..., r_M - r'_M)$ is a permutation of an element in

$$\{(0,0,...,0),(1,-1,0,...,0),(1,0,0,...,0),(-1,0,0,...,0)\}.$$

<u>Definition 4.1</u> The pair $(H, \tilde{H})$ is an A_M-Leonard pair on the domain $\mathcal{D}$ if the following conditions (i) to (vii) are satisfied.

(i) H is an M-dimensional subspace of $End(V)$ whose elements are diagonalizable and mutually commute.
(ii) $\tilde{H}$ is an M-dimensional subspace of $End(V)$ whose elements are diagonalizable and mutually commute.
(iii) There exists a bijection $\alpha \mapsto V_\alpha$ from $\mathcal{D}$ to the set of common eigenspaces of H such that for all $\alpha \in \mathcal{D}$,

$$\tilde{H} V_\alpha \subset \sum_{\beta \in \mathcal{D},\ \beta\ adj\ \alpha} V_\beta .$$

(iv) There exists a bijection $\alpha \mapsto \tilde{V}_\alpha$ from $\mathcal{D}$ to the set of common eigenspaces of H such that for all $\alpha \in \mathcal{D}$,

$$H \tilde{V}_\alpha \subset \sum_{\beta \in \mathcal{D},\ \beta\ adj\ \alpha} \tilde{V}_\beta .$$

(v) There does not exist a subspace W of V such that $HW \subset W, \tilde{H} W \subset W, W \neq 0, W \neq V$.
(vi) Each of $V_\alpha, \tilde{V}_\alpha$ has dimension 1 for $\alpha \in \mathcal{D}$.
(vii) There exists a nondegenerate symmetric bilinear form $\langle , \rangle$ on V such that both

$$\langle V_\alpha, V_\beta \rangle = 0, \text{ if } \alpha, \beta \in \mathcal{D}, \quad \langle \tilde{V}_\alpha, \tilde{V}_\beta \rangle = 0, \text{ if } \alpha, \beta \in \mathcal{D},$$

I think there is still a room of discussion what is the most reasonable definition of A_M-Leonard pair. For example, in the above definition, $\mathcal{D} = \mathcal{D}^*$ is assumed. Should we also consider the case of $\mathcal{D} \neq \mathcal{D}^*$.

<u>Definition 4.2</u>. A symmetric association scheme $\mathfrak{X}$ is called an A_M multivariate P-polynomial or Q-polynomial association scheme on $\mathcal{D} = \{\alpha \in \mathbb{N}^M \mid |\alpha| \leq N\}$ if the following conditions are satisfied.
(a) $\mathfrak{X}$ is an M-variate P-polynomial association scheme on $\mathcal{D}$ for some monomial order $\leq_1$ such that for $\alpha \in \mathcal{D}$ and $i = 1, 2, ..., M$, if $p^{\beta}_{e_i,\alpha} \neq 0$ then β is adjacent to α.
(b) $\mathfrak{X}$ is an M-variate Q-polynomial association scheme on $\mathcal{D}$ for some monomial order $\leq_2$ such that for $\alpha \in \mathcal{D}$ and $i = 1, 2, ..., M$, if $q^{\beta}_{e_i,\alpha} \neq 0$ then β is adjacent to α.

<u>Theorem 4.3</u>. An A_M multivariate P- and Q-polynomial association scheme on $\mathcal{D} = \{\alpha \in \mathbb{N}^M \mid |\alpha| \leq N\}$ has the structure of A_M-Leonard pairs for $\mathbb{F} = \mathbb{C}$.

<u>Corollary 4.4</u>. If the domain $\mathcal{D}$ of a non-binary Johnson association scheme and the association scheme obtained from attenuated space becomes an isosceles right triangle, then each of these association scheme has the structure of A_2-Leonard pair.

Factored A_2-Leonard pairs

Crampé-Zaimi: Factored A_2-Leonard pair (ArXiv:2312.08312v3), considers the following very restricted class of A_2-Leonard pairs. We will not discuss the details of this definition, but the essence of it is that the spherical functions are expressed as a product of two of one-variate Askey-Wilson orthogonal polynomials, resulting so called Tratnik type bivariate orthogonal polynomials.

The non-binary Johnson association schemes and attenuated spaces are examples of factored A_2-Leonard pairs, if the domain $\mathcal{D}$ is an isosceles right triangle. For example, the Gelfand pair $(W(C_n), W(C_n)_J)$, where $J = \{1, 2, ..., n\} - \{i\}(2 \leq i \leq n-2)$ gives a factored A_2-Leonard pair. This Gelfand pair (commutative association scheme) is in fact imprimitive. We can see that a factored A_2-Leonard pair must be imprimitive. On the other hand, the corresponding Gelfand pair (G, G_J) (where G is the Chevalley group of type C_n and G_J is the parabolic subgroup corresponding to J is primitive. So, (G, G_J) cannot be a factored A_2-Leonard pair. (It seems that (G, G_J) is in fact bivariate P-and Q-polynomial association scheme, but this is still yet to be proved.) Anyway, the spherical functions are not expressed by Tratnik type.

Conclusions

It would be very important to try to get the multivariate version of Leonard Theorem, namely to get the spherical functions (as well as character tables) of multivariable P-and Q-polynomial association schemes, or to determine higher rank Leonard pairs. As we have discussed, this is not easy at all, and we still do not have a good picture. As we discussed already, these spherical function include those of symmetrizations of (commutative) association schemes or those of association schemes coming from isotropic subspaces. Perhaps, there should exist more such association schemes and multivariable orthogonal polynomial, far beyond the already known examples. There are many more candidates for possible multivariable P-and Q-polynomial association schemes. These theories have just been started, and I do hope we can develop these study furthermore in this direction.

Thank You

Appendix: Bivariate P-polynomial association scheme of type (α, β).

Let $0 \le \alpha \le 1$ and $0 \le \beta < 1$. We define the (α, β)-order on the monomials.

$$x^m y^n \le_{\alpha,\beta} x^i y^j, \text{ if } m + \alpha n \le i + \alpha j \text{ and } \beta m + n \le \beta i + j.$$

(We also identify $x^i y^j$ with $(i, j) \in \mathbb{N}^2$.)

Definition. Let $\mathcal{D} \subset \mathbb{N}^2$. Let α, β, and $\le_{\alpha,\beta}$ be as before. Then the association scheme $\mathfrak{X}$ is called a bivariate P-polynomial association scheme of type (α, β) on the domain $\mathcal{D}$ if the following conditions are satisfied.

(i) There exists a relabeling of adjacency matrics:

$$\{A_0, A_1, ..., A_d\} = \{A_{mn} \mid (m, n) \in \mathcal{D}\}$$

such that for $(i, j) \in \mathcal{D}$,

$$A_{ij} = v_{ij}(A_{10}, A_{01}),$$

where $v_{ij}(x, y)$ is (α, β)-compatible bivariate polynomial of degree (i, j). (Namely, a bivariate polynomial $v(x, y)$ is called (α, β)-compatible, if the monomial $x^i y^j$ appears and all other monomials appearing are smaller than $x^i y^j$ for the order $\le_{\alpha,\beta}$.)

(ii) $\mathcal{D}$ is (α, β)-compatible. (Namely, if $(i, j) \in \mathcal{D}$ and $(m, n) \le_{\alpha,\beta} (i, j)$ implies $(m, n) \in \mathcal{D}$.)