G2C2 Lecture No. 6. A personal view on Gelfand pairs and commutative association schemes

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Basic References

[1] Bannai-Bannai-Ito-Tanaka: Algebraic Combinatorics, Chapter 5, De Gruyter, 2021.

[2] Bannai-Bannai-Tanaka-Zhu: Design Theory from the viewpoint of algebraic combinatorics, Graphs and Comb. 31 (2017), 1-41.

We will give more references later.

Spherical designs of harmonic index T

In this section, we consider another generalizations of spherical t-designs. (Later we will consider also for designs on association schemes.)

Definition (Spherical designs of harmonic index T) Let T be a subset of $\{1, 2, \ldots\}$. Let Y be a finite non-empty subset of S^{n-1}. Then Y is called a spherical design of harmonic index T (or spherical T-design), if

$$\sum_{x \in Y} f(x) = 0 \text{ for all } f \in \mathrm{Harm}_k(\mathbb{R}^n), k \in T.$$

The case $T = \{1, 2, \ldots, t\}$ corresponds the usual spherical t-designs on S^{n-1}. The case $T = \{t\}$ is called a spherical design of harmonic index t.

Let $Q_{n,k}(x)$ be the Gegenbauer polynomial of degree k defined before. Then we have

$$Q_{n,k}(1) = \dim\Big(\mathrm{Harm}_k(\mathbb{R}^n)\Big) = \binom{n-1+k}{k} - \binom{n-1+k-2}{k-2}.$$

Bannai-Okuda-Tagami (2015) gave the Fisher type inequality for spherical designs of harmonic index $2e$ as follows.

Theorem. Let Y be a spherical design of harmonic index $2e$. Then

$$|Y| \geq 1 + \frac{Q_{n,2e}(1)}{c_{n,2e}} := b_{n,2e},$$

where

$$c_{n,2e} = -\min_{-1\leq x\leq 1} Q_{n,2e}(x). \qquad (*)$$

Equality holds if and only if $Q_{n,2e}(\alpha) = -c_{n,2e}$ for any $\alpha \in A(Y)$, where

$$A(X) = \{x \cdot y \mid x, y \in Y,\ x \neq y\}.$$

We say Y is tight if the equality hold in the above formula (*).

More generally, Zhu-Bannai-Bannai-Kim-Yu (Electron. J. Comb. 2017) dealt with the case when T consists of ℓ positive even integers $t_1 > t_2 > \cdots > t_\ell \geq 2$. They considered a test function

$$F(x) = f_0 + \sum_{k\in T} f_k Q_{n,k}(x)$$

with the normalization $f_{t_1} = 1$ so that the following condition is satisfied:

$F(x)$ is non-negative and has ℓ non-negative zeros on $[-1, 1]$.

Theorem. Let Y be a spherical design of harmonic index $T = \{t_1 > t_2 > \cdots > t_\ell\}$. Then we have

$$|Y| \geq \frac{F(1)}{f_0}.$$

If there is such a function $F(x)$, a lower bound for $|Y|$ is obtained. However, it is generally open what is the best (optimal) $F(x)$. Even in the case of usual tight spherical $2e$-designs, this is not easy (i.e., not yet answered). So, I believe the definition of tight spherical t-design is a kind of conventional nature. Nonetheless, I believe that the usual definition of tight spherical t-design is natural and meaningful. (So, it is in general not easy to define the concept of tight T-design for general T, in particular when ℓ is large. (Still we consider natural lower bound for some reasonable T, and studying the case where the lower bound is attained (tight T-designs) even we know that this definition of tight T-design is conventional. (See Zhu-Bannai-Bannai-Kim-Yu(2017).)

Anyway, we studied tight spherical harmonic index $2e$-designs. In Okuda-Yu ((2016) these tight spherical designs of harmonic index 4 are completely classified, and further studies were made in Bannai-Okuda-Tagami (2015). One interesting result there is that for each e there is a constant c_e such that if there is a harmonic index $2e$-design Y on S^{n-1}. Then $|Y| \geq c_e n^e$ if n goes to ∞. This implies that the size of a harmonic index $2e$-design is basically the same order as the Fisher type lower bound of usual spherical $2e$-designs.

Harmonic index t-design in $H(n, 2)$.

We can consider T-design where T is a subset of $\{1, 2, ..., d\}$ (the set of indices of the primitive idempotents) in $H(n, 2)$, or more generally in any commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$.

Definition. Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative association scheme. If a subset Y of X satisfies the condition that $E_i \phi_Y = 0$ for any $i \in T$, then Y is called a T-design in $\mathfrak{X}$. (Note that the condition $E_i \phi_Y = 0$ is equivalent to the condition that $a_1^* = a_2^* = \cdots = a_t^* = 0$ for the dual distribution of Y. Also, note that if $T = \{1, 2, \ldots, t\}$ then a design of harmonic index T is a usual t-design in $\mathfrak{X}$. If $T = \{t\}$, then it is called a harmonic index t-design.)

We have similar result as in spherical designs of harmonic index t. Namely, in $H(n, 2)$ for each e there is a constant c_e such that if there is a harmonic index $2e$-design Y on S^{n-1}, Then $|Y| \geq c_e n^e$ if n goes to ∞. Also, the concept of tight t-designs of harmonic index T is defined (for some T but not for all T), and the existence of tight designs of harmonic index T was studied for some T.

Recall that we have already considered the following generalizations of spherical t-designs and t-design in a Q-polynomial association scheme.
(i) Two step generalization to Euclidean t-design on p concentric spheres (for the case of spherical t-designs, see Lecture 4) and two step generalization to relative t-designs on p shells of the Q-polynomial association scheme (for the case of t-design on Q-polynomial association schemes, see Lecture 5).

As we just discussed there was the following generalization
(ii) Consider T-design instead of t-design, i.e., harmonic index T-design. (We were interested in Fisher type lower bound as well as the classification of tight T-designs.)

Another generalization is
(iii) To change the basic space, namely instead of sphere S^{n-1} we consider other spaces. Most natural ones are replace the sphere by various topological spaces M, in particular rank one compact symmetric spaces, or furthermore compact symmetric spaces of any rank. (Or more generally compact Gelfand pairs coming from Lie groups.) While, instead of Q-polynomial association scheme, we consider more general commutative association schemes, or more specifically so called Gelfand pairs (multiplicity free finite permutation groups).

So, I believe that our tentative final target is try to classify basic spaces, i.e., Gelfand pairs and commutative association schemes.) Now, we want to discuss on this problem. (Neither technically nor rigorously, but in a way like waving hands.)

The following part is based on my talk at at Wokshop on Algebraic Combinatorics, at Academia Sinica, Taipei, Jan. 26, 2022.

(Abstract). The classification problem of P-and Q-polynomial association schemes has been (and still is) a central problem in algebraic combinatorics, since around 1980. Beyond that, we have been interested in studying more general commutative association schemes and finite Gelfand pairs, along the line of study of P-and Q-polynomial association schemes.

See for example, E. Bannai: Character tables of commutative association schemes, in Finite Geometries, Buildings, and Related Topics (ed. by W. M. Kantor et al.), Oxford Univ. Press, 1990, 105-128.

In this talk, I will try to present what I know and also what I want to know on commutative association schemes and finite Gelfand pairs. Sorry that this part of my lecture is only from my very personal view, and may be too personal.

Conceptually, it seems that there are close connections between the concept of compact symmetric spaces of rank one and the concept of P-and Q-polynomial association schemes. This was an important motivation why I wanted to study P-and Q-polynomial association schemes in the first place.

Please recall D. Leonard's theorem (1982) and the book Bannai-Ito: Algebraic Combinatorics, I (Benjamin/Cummings,1984).
This semester (in fall 2021), I gave a course of Taiwan Mathematical School at NTU. In the first half, I lectured on Chapter 6 of new book of Bannai-Bannai-Ito-Tanaka: Algebraic Combinatorics (De Gruyter, 2021). Chapter 6 treats P-and Q-polynomial association schemes and the classification of Leonard pairs. In the second half of the course, I lectured on the character tables of many examples of finite Gelfand pairs (G, H) (multiplicity-free transitive permutation groups) and commutative association schemes, emphasizing the finite group theoretical viewpoint.

Then I began to realize that some of my understandings were not precise enough and that the relation between general Gelfand pairs (G, H) in Lie groups and Gelfand pairs (G, H) in finite groups is very delicate and not so simple as I originally imagined. So, I want to present some of my new understandings, together with my reflections.

First, let us recall that compact symmetric spaces were classified by E. Cartan (1926). Compact symmetric spaces are Gelfand pairs. There is a weaker concept called "weakly symmetric spaces" defined by Selberg (1956) and are also shown to be Gelfand pairs. Also, weakly symmetric spaces are classified (Akhiezer-Vinberg, 1999; Nguyen, 2000). On the other hand, (under some mild additional conditions such as compact, connected, G being a simple Lie group), it is shown that compact Gelfand pairs were already classified (cf. Krämer, 1979). So, we basically have the list of compact Gelfand pairs (for Lie groups). I was not aware of this fact before. The list (essentially the same list as the compact symmetric or weakly symmetric spaces) is very interesting.

Question. For each symmetric space, or weakly symmetric space, or Gelfand pair (G, H), is there any corresponding finite Gelfand pair (G', H') ?

In many cases yes, in the sense that we can find some candidates. On the other hand, it is not always easy to see that when H' actually becomes a multiplicity-free subgroup of G'. (There are many things left undecided.)

The maximal subgroups of finite simple groups are now classified to some extent. (This classification is not complete even if we restrict to multiplicity-free maximal subgroups.)

- For alternating groups, by Saxl and others,
- For classical groups, by Aschbacher and many others,
- For exceptional Lie type groups, by Cooperstein, Kleidman, Craven (arXiv;2103.04869v2) and others,
- For sporadic simple groups, by Ivanov and others.

Here are some examples of Gelfand pairs (G, H) for finite classical groups. After pioneering work by Gow(1984) and Inglis-Liebeck-Saxl (1986), Inglis in his Ph. D thesis (1986) gives all the candidates of maximal subgroups which are multiplicity-free for finite classical Lie type groups. For example, for type (A), the follow are the examples.
$G = GL(n, q^2), H = GL(n, q)$,
$G = GL(n, q^2), H = GU(n, q^2)$,
$G = GL(2n, q), H = Sp(2n, q)$,
$G = GL(2n, q), H = GL(n, q^2)$.

Also, similar results were basically obtained for other classical Lie type groups.

For example, it is seen that $(Sp(4, q), Sz(q))$ (for q being an odd power of 2) is a Gelfand pair, but $(Sp(4, q^2), Sp(4, q))$ is not a Gelfand pair but almost multiplicity-free. Namely, there is exactly one irreducible character appearing with multiplicity 2 in $(1_H)^G$, and other irreducible representations are all multiplicity at most 1.

There are four main sources to get finite Gelfand pairs and commutative association schemes.

(i) G is a group (of Lie type) acting irreducibly on a vector space V (over a finite field $\mathbb{F}_q$). Let V_0 be a subspace of V, and let H be the stabilizer of V_0 in G. Then consider the action of G on $X = G/H$.

(ii) G is a group (of Lie type) acting irreducibly on a vector space V (over a finite field $\mathbb{F}_q$). Let H be also an irreducible subgroup of G. Then consider the action of G on $X = G/H$.

(iii) Let G be a group, and let $G \times G$ acts on the diagonal subgroup diag(G). Namely, we consider the group association scheme $\mathcal{X}(G)$ of G. Then we get a Gelfand pair $(G \times G, \mathrm{diag}(G))$. Note that there is a well-known correspondences between the character table T of the group G and the first eigenmatrix (character table) P of the group association scheme $\mathcal{X}(G)$.

(iv) Let H be a subgroup of $GL(V)$ and let $G = V.H$ (semi-direct product) acts on the coset $V = G/H$. (This is always a Gelfand pair.)

I think it is still true that these four cases are the important sources to get finite Gelfand pairs and commutative association schemes.

In a series of papers joint with Sung-Yell Song, Hao Shen, Hongzeng Wei, Wing Man Kwok, Noriaki Kawanaka, Hajime Tanaka, Hiromichi Yamada, and others, I studied the theme that many character tables of commutative association schemes are controlled by the character table of smaller association schemes. This was started with the work on the character tables of simple Moufang loops of Paige with Song (Proc. LMS,1988), and then progressed very successfully. For the character table of the action of a Chevalley group on the isotropic subspaces (namely the cosets by a parabolic subgroup), it was earlier known to be controlled by the character table of the action of the Weyl group on its cosets by a Weyl subgroup corresponding to the parabolic subgroup (Curtis-Iwahori-Kilomyer, 1970, etc.). We studied the action on non-isotropic subspaces for small dimensions (Bannai-Song-Shen-Wei, 1990, etc.) as well as the cases $GL(2n,q)/Sp(2n,q)$ (Bannai-Kawanaka-Song, 1990), $GL(2n,q)/GL(n,q^2)$ (Bannai-Tanaka, 2003), $G_2(q)/O_6^{\epsilon}(q)$ (Bannai-Song-Yamada, 2008), and so on.

Now, I would like to reflect on these work.

(1) We have not yet completely determined when a subgroup H which fixes a non-isotropic subspace is multiplicity-free. (We considered the cases where the dimension of the non-isotropic subspace is of dimension 1 (orthogonal and unitary cases) and dimension 2 (symplectic case) and assumed that other cases are probably not multiplicity-free. I hope that answering to this remaining question is possible. Even if most of them are not exactly multiplicity-free, the multiplicity of irreducible characters appearing in the permutation character seems to be bounded. So, there are some cases where the situation is very close to the multiplicity-free case, even if it is not exactly multiplicity-free. I now believe that we better study the cases that are close to multiplicity-free.

(2) Another reflection is that we wanted to approach the classification problem of finite Gelfand pairs, based on the analogy of the classification of compact symmetric spaces, or compact Gelfand pairs. As the first approximation, it looked promising. However, the details are far more complicated than I originally thought.

The most natural (compact) symmetric spaces coming from classical Lie groups are:

$G = SU(n+m), H = S(U(n) \times U(m))$

($G = SU(n+m), H = SU(n) \times SU(m)$ is a weakly symmetric space.)

$G = SO(n+m), H = SO(n) \times SO(m)$,

$G = Sp(n+m), H = Sp(n) \times Sp(m)$.

It is known that for each finite classical simple group, there is a good maximal parabolic subgroup that gives a P-and Q-polynomial structure. However, this does not correspond to the above compact Gelfand pair (G, H). For each above (G, H) there is a natural (G', H') that can naturally be expected to give a finite Gelfand pair. But, actually, that subgroup H' is the stabilizer of a non-isotropic subspace, and far different from the parabolic subgroup that gives a P-and Q-polynomial structure. (Also, more importantly, this does not give a Gelfand pair in general, in particular if n is not very small and $n \leq m$, as I mentioned already.)

I originally thought naively that there should be a good multivariable version of Askey-Wilson polynomials, and many of the character tables of finite Gelfand pairs as well as commutative association schemes should be described by using them. Also, I thought that P-and Q-polynomial association schemes of rank ℓ may be naturally defined. However, this was no so simple at all, although there are some multivariable versions of Askey-Wilson polynomials are known. So, here are some of my deep reflections.

(a) There are not so many examples that can be considered as a P-and Q-polynomial association schemes of rank 2 (or higher arbitrary rank ℓ), both in compact symmetric space case and in finite case. (This was at the point in my talk at that time, January 2022.) As mentioned in the previous page of the slides, if $n \leq m$ and $n \geq 2$ then the natural corresponding finite case seems not to be even Gelfand pair at all, although the multiplicities are bounded by small number (in other words, near multiplicity-free). Some of compact symmetric space case, such as $G = SU(n), H = SO(n)$, give a finite Gelfand pair, say $G = GL(n, q^2), H = GL(n, q)$, but this Gelfand is far from P-and Q-polynomial schemes. It is difficult to interpret this as higher rank P-and Q-polynomial scheme in some reasonable way, so far. It would be very desirable, if we could regard the group association scheme $\mathfrak{X}(GL(n, q))$ or $\mathfrak{X}(GU(n, q))$ (that is a controlling association scheme of the finite Gelfand pair $(GL(n, q^2), GL(n, q))$ as a P-and Q-polynomial association scheme of higher rank, say $n-1$(or n?), but perhaps it is not so easy. Also, it would be desirable, if $(GL(2n, q), GL(n, q^2))$ could be regarded as a P-and Q-polynomial association scheme of rank $[\frac{n}{2}]$. (Also, it would be interesting, if $(Sp(2n, q), GL(n, q))$ or $(Sp(2n, q), GU(n, q))$ could be regarded as a P-and Q-polynomial association scheme of rank n.)

(b) It seems that the only known good examples of P-and Q-polynomial association schemes of rank 2 (or higher arbitrary rank ℓ.) are basically those obtained as wreath product $(G \text{ wr } S_n)/(H \text{ wr } S_n)$ found by Mizukawa (2004) and Mizukawa-Tanaka (2004), where (G, H) is a Gelfand pair. Note that several multivariable versions of Askey-Wilson polynomials are already known and some of them are interpreted as the spherical functions of quantum groups (say), but not necessarily of commutative association schemes.

(c) In their paper: Iliev-Terwilliger (2012), they interpret these multivariable version of Askey-Wilson polynomials in a very visual way as higher rank Leonard systems. I am very much fascinated with the Problem 7.1 proposed in Iliev-Terwillige (2012). They regard their polynomials related to the root system of type A_n and ask whether there are multivariable polynomials corresponding to other simple root systems. On the other hand, I am not certain whether there are any examples of Gelfand pair (or commutative association schemes) associated to these multivariable orthogonal polynomials corresponding to other root systems (although I wish they could exist).

Then, how should we proceed ?

I have no definite answer. But, I would like to study the structure of the Bose-Mesner algebra of controlling (small) association schemes carefully. In particular, those controlling (small) association schemes that appear in our series of papers (mentioned at the beginning of my talk), such as

$\mathfrak{X}(PGL(2,q)), \mathfrak{X}(PSL(2,q))$, $PGL(2,q)/D_{2(q-1)}$, $PGL(2,q)/D_{2(q+1)}$,
$PGL(2,q)/Z_{q+1}$, $PGL(2,q)/Z_{q-1}$, $O(3,q)/O^{+}(2,q)$, **etc. etc.**

I believe that more mechanisms are hidden there.

Conclusions. **(i) More group theoretical study should be given on the classification of finite Gelfand pairs. Personally, I was interested in studying Gelfand pairs (commutative association schemes) mostly avoiding to use the classification of finite simple groups. Note that there are many papers that study the structures of maximal subgroups using the classification of finite simple groups, by O'Nan, Scott, Cameron, Praeger, Liebeck, Saxl, Seitz, Aschbacher, Ivanov, and many many others. Still many materials are left open, even on the classification of maximal subgroups, or more specifically the classification of the multiplicity-free maximal subgroups (Gelfand pairs), as well as the determination of their character tables (spherical functions). I think there are some advantages and some disadvantages in trying to avoid the classification of finite simple groups. I do not have any serious regret on this attitude of mine. (The life is limited, and so we have to choose what we can do.) However, I think I could have learned more on group theoretical results and techniques that use the classification of finite simple groups.**

I just become aware that there are many studies that apply O'Nan-Scott theorem (on the structure of maximal subgroups) to various special cases: to the classifications of doubly transitive groups, rank 3 permutation groups (Liebeck), distance-transitive graphs (Praeger-Saxl-Yokoyama), multiplicity-free subgroups, namely Gelfand pairs (Baddeley, J. Algebra, 1993), etc. etc.. I think we should now look at many explicit examples of Gelfand pairs from this viewpoint of group theory.

(ii) I still believe in what I have been working on algebraic combinatorics. I still believe that the classification problem of P-and Q-polynomial association schemes is very important, in a sense as the classification of finite simple groups is obviously and extremely important. Also, the classification problem of general (primitive) commutative association schemes are naturally very important, although it may not be intractable. I think even the special case: the classification of maximal subgroups that are multiplicity-free (so, primitive Gelfand pairs) is important, even it depends on the classification of finite simple groups. At the same time, I feel it is very pity that I (as well as most of us) cannot understand the full details of the proof of the classification of finite simple groups. Now, there are many many proofs (in mathematics general) that we cannot understand the details, partly because the techniques are so complicated, and also extensive computer calculations and/or AI are involved. Even so, I hope mathematics can be survived in the future.

To be continued.