# G2C2 Lecture No. 5. Relative $t$-designs in association schemes.

**Eiichi Bannai**

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**Basic Reference**

**[1] Desarte: Pairs of vectors in the space of an association schemes, Philips Res. Rep. (1977).**

**[2] Bannai-Bannai-Tanaka-Zhu: Design Theory from the viewpoint of algebraic combinatorics, Graphs and Comb. 31(2017), 1-41.**

**We will give more references later.**

# The concept of relative $t$-designs on Q-polynomial association schemes

As Euclidean $t$-designs are obtained as two step generalization of spherical $t$-designs, relative $t$-designs are obtained as a similar two step generalization of combinatorial $t$-designs in the following way. This definition is due to Delsarte (1977) and historically, the concept of relative $t$-designs existed earlier than that of Euclidean designs (Neumaier-Seidel, 1988). We even suspect that relative $t$-designs may have served as a hint to define Euclidean $t$-designs. On the other hand, the theory of Euclidean $t$-designs is easier to handle, i.e., the theory of harmonic analysis is more beautiful and easier than that of association schemes, so the theory of Euclidean $t$-designs was ahead of the theory of relative $t$-designs. Then the theory of relative $t$-designs have started later, in a sense pursuing the analogy of the theory of Euclidean $t$-designs.

We first give the general definition.

**Definition** (Relative $t$-designs in Q-polynomial association scheme, Delsarte 1977).
Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme. Fix $x_0 \in X$.
Let $Y$ be a subset of $X$, and let $w : Y \to \mathbb{R}_{>0}$ be a weight function on $Y$.
The pair $(Y, w)$ is called a relative $t$-design on $\mathfrak{X}$ with respect to $x_0$ if

$$E_i \phi_{(Y,w)} \in \langle E_i \phi_{\{x_0\}} \rangle \quad \text{for all } i = 1, 2, \ldots, t.$$

Here $\phi_{(Y,w)}$ is the characteristic function of $Y$ defined by

$$\phi_{(Y,w)} = w(x) \text{ if } x \in Y \text{ and } \ = 0, \text{ otherwise }.$$

For a fixed point $x_0 \in X$, let $X_i = \{x \in X \mid (x_0, x) \in R_i\}$ for $i = 0, 1, \ldots, d$ and $Y \subset X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$.

An equivalent definition of a relative $t$-design is also given as follows, in the form of a cubature formula.

Definition. The condition of a relative $t$-design (for the Q-structure) in the above definition is equivalent to the following condition.

$$\sum_{\nu=1}^{p} \frac{W(Y \cap X_{r_\nu})}{|X_{r_\nu}|} \sum_{x \in X_{r_\nu}} f(x) = \sum_{x \in Y} w(x) f(x)$$

for any

$$f \in L_0(X) \perp L_1(X) \perp \cdots \perp L_t(X)$$

where $L_j(X) \subset \mathbb{R}^X$ is the $j$th eigenspace of the Bose-Mesner algebra (onto which $E_j$ acts as the orthogonal projection), and $W(Y \cap X_{r_\nu}) = \sum_{y \in Y \cap X_{r_\nu}} w(y)$.

We note that the concept of relative $t$-designs for a P-polynomial association scheme is also defined (see later).

Now we want to discuss relative $t$-designs in the binary Hamming association scheme $H(n,2)$. It is known that relative $t$-designs in $H(n,2)$ are equivalent to regular $t$-wise balanced designs as follows.

Definition (Regular $t$-wise balanced designs). Let $V$ be a finite set with cardinality $v$. Let $\mathcal{B}$ be a subset of $2^V$ and assign a weight function $w:\mathcal{B}\to\mathbb{R}_{>0}$. Then the triple $(V,\mathcal{B},w)$ is called a $j$-wise balanced design if there exists $\lambda_j$ satisfying:

$$\sum_{B\in\mathcal{B},Z\subset B} w(B)=\lambda_j \text{ for all } Z\in\binom{V}{j}.$$

Moreover, $(V,\mathcal{B},w)$ is a regular $t$-wise balanced design, if the above equality holds for all $j=1,2,\ldots,t$.

Note that when we consider relative $t$-design in $H(n,2)$ as a regular $t$-wise balanced design, we are considering $x_0$ to be the vector $(0,0,\ldots,0)$ so that $X_j$ naturally corresponds to $\binom{V}{j}$. As in the case of ordinary combinatorial $t$-designs, we consider (mainly ?) non-weighted designs, but also weighted designs.

When we consider weighted combinatorial $t$-designs, the values of the weight are positive real numbers, unless otherwise stated. It should be remarked that a $t$-wise balanced design is not necessarily a $(t-1)$-wise balanced design.

For a regular $2e$-wise balanced design in $H(n,2)$, i.e., a relative $2e$-design on $H(n,2)$ on $p$ shells, Ziqing Xiang (2012) proved the following Fisher type lower bound:

$$|Y| \geq m_e + m_{e-1} + \cdots + m_{e-p+1},$$

where $m_i = \binom{v}{i} - \binom{v}{i-1}$.

We conjecture that in a general case, we have the Fisher type lower bound for relative $2e$-designs on $p$ shells in Q-polynomial associationschemes under some reasonable assumptions:

$$|Y| \geq m_e + m_{e-1} + \cdots + m_{e-p+1},$$

where $m_i = \text{rank}(E_i)$.

Next, we introduce the concept of relative $t$-design in a P-polynomial association scheme, generalizing the concept defined for $H(n,2)$ in Delsarte-Seidel (1989).

For $z \in X$, we define a real valued function $f_z$ on $X$ by:

$$f_z(x) = 1, \text{ if } x \in X_i, i \geq j \text{ and } (x,z) \in R_{i-j}, \text{ ; and } = 0, \text{ otherwise .}$$

We let $\mathrm{Hom}_j(X) = \langle f_z \mid z \in X_j \rangle$ and give the following definition.

<u>Definition</u> (Relative $t$-design in P-polynomial association scheme). Let $\mathfrak{X}$ be a P-polynomial association scheme of class $d$. A weighted subset $(Y,w)$ is a relative $t$-design for the P-polynomial structure of $\mathfrak{X}$ with respect to $x_0$, if

$$\sum_{\nu=1}^{p} \frac{W(Y \cap X_{r_\nu})}{|X_{r_\nu}|} \sum_{x \in X_\nu} f(x) = \sum_{y \in Y} w(y) f(y)$$

for any $f \in \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_t(X)$.

Bannai-Bannai-Suda-Tanaka (2015) made the following observation.

<u>Proposition</u>. Le $\mathfrak{X}$ be the Hamming association scheme $H(n,q)$. Then

$$\mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_t(X) = L_0(X) + L_1(X) + \cdots + L_t(X)$$

for $t = 1, 2, ..., n$.

This implies that relative $t$-design in $H(n, q)$ for the P-structure and the Q-structure are equivalent. On the other hand, $H(n, q)$ is essentially the only P-and Q-polynomial association schemes with this property holds. (Bannai-Bannai-Suda-Tanaka, Electron. J. Comb. 2015)

Consider the binary Hamming association scheme $H(n, 2)$. Then its $i$th shell $X_i$ with respect to $x_0 = (0, 0, ..., 0)$ induces the Johnson association scheme $J(n, i)$. In view of Delsarte's interpretation (1977) mentioned above, it follows that $Y \subset X_i$ is a relative $t$-design on $H(n, 2)$ if and only if $Y$ is a (combinatorial) $t$-design on $X_i = J(n, i)$.

Motivated by this observation, we propose the following definition.

Definition. Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a Q-polynomial association scheme, and fix $x_0 \in X$. We define a subset $Y \subset X_i$ to be a $t$-design on $X_i$, if $Y$ is a relative $t$-design on $\mathfrak{X}$ with respect to $x_0$.

Then a natural problem is whether or not it is possible to identify the $t$-design on $X_i$, for each of the association schemes in the core list of P-and Q-polynomial association schemes, as harmonic index $T$-designs on $X_i$ for some subset $T$ of an index set of primitive idempotents of $X_i$, when $X_i$ is commutative. We consider this question for some special cases.
(Note that for a subset $T \subset \{1, 2, ..., d\}$, $Y \subset X$ is a (harmonic index) $T$-design, if $E_i \phi_Y = 0$ for any $i \in T$.)

## Algebraic Definition of $t$-Design on One Shell in $J(v, d)$.

Consider the Johnson association scheme $J(v, d)$. Observe that the $i$th shell $X_i$ is the product of $J(d, i)$ and $J(v - d, i)$. Thus, if $\{E'_0, E'_1, ..., E'_i\}$ (resp. $\{E''_0, E''_1, ..., E''_i\}$) are the primitive idempotents of $J(d, i)$ (resp. $J(v - d, i)$), then the primitive idempotents of $X_i$ are given by $\{E_{k,\ell} \mid k, \ell = 0, 1, ..., i\}$, where

$$E_{k,\ell} = E^{(i)}_{k,\ell} := E'_k \otimes E'_\ell (k, \ell = 0, 1, ..., i).$$

It follows that the $t$-design in $X_i$ are then precisely the harmonic index $T$-design on $X_i$ with $T = \{(k, \ell) \mid 0 < k + \ell \leq t\}$. (Martin 1998,1999).

## Algebraic Definition of $t$-Design on One Shell in $H(d, q)$.

In this case the $i$th shell $X_i$ can be regarded as a kind of product of $H(i, q - 1)$ and $J(d, i)$, although this is not exactly the product scheme. Dunkl (1976) studied the harmonic analysis in detail, and determined the set of primitive idempotents of the commutative association scheme $X_i$ (see also Tarnanen-Aaltonen-Goethals, 1985). Namely, the set of primitive idempotents corresponds to the irreducible decomposition of permutation representation on $X_i = H/H_i$, where $G = (S_q)^d \cdot S_d$ (the automorphism group of $H(d, q), H = G_{x_0} = (S_{q-1})^d \cdot S_d$ and $H_i = ((S_{q-2})^i \cdot S_i) \times ((S_{q-1})^{d-i} \cdot S_{d-i})$. If we assume $i << d$ for simplicity, then the parameter set of the primitive idempotents is given by $\{i, \ell, m\}$ with $0 \leq \ell \leq i$, and $t$-designs in $X_i$ are identified with harmonic index $T$-designs on $X_i$ with $T = \{(\ell, m) \neq (0, 0) \mid 0 \leq \ell \leq m \leq t\}$.

# Relative $t$-designs and Assmus-Mattson Theorem.

Here is a generalization of Assmus-Mattson theorem in the context of Delsarte [Pairs of vectors... (1977)].

Theorem. (Delsarte, 1977). Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme. Fix $x_0 \in X$ and let $X_i = \{x \in X \mid (x_0, x) \in R_i\}$. Let $(Y, w)$ be a $t$-desgn in $\mathfrak{X}$ on a union of $p$ shells, i.e., $Y \subset X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$. Let $Y_{r_\nu} = Y \cap X_{r_\nu} (\nu = 1, 2, ..., p)$. Then each $(Y_{r_\nu}, w_{|Y_{r_\nu}})$ is a relative $(t - p + 1)$-design on $\mathfrak{X}$ with respect to $x_0$.

Here we remark that the assumption in the above theorem is weakened to the condition that $Y$ is a relative $t$-design (instead of a $t$-design).

Theorem. (Tanaka, 2009). Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme. Fix $x_0 \in X$ and let $X_i = \{x \in X \mid (x_0, x) \in R_i\}$. Let $(Y, w)$ be a relative $t$-design in $\mathfrak{X}$ on a union of $p$ shells, i.e., $Y \subset X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$. Let $Y_{r_\nu} = Y \cap X_{r_\nu} (\nu = 1, 2, ..., p)$. Then each $(Y_{r_\nu}, w_{|Y_{r_\nu}})$ is a relative $(t - p + 1)$-design on $\mathfrak{X}$ with respect to $x_0$.

Hajime Tanaka: New proofs of of the Assmus-Mattson theorem based on Terwilliger algebras, Europ. J. Comb. (2009)

## Tight relative $t$-designs on $H(n, 2)$ and $J(v, k)$.

There are some works to study tight relative 2-designs on $H(n, 2)$, in particular the case of on two shells. (On one shell case is tight 2-design, i.e., symmetric design on $J(v, i)$.)

Bannai-Bannai-Bannai: On the existence of tight relative 2-designs on binary Hamming association schemes, Discrete Math. 2014,
Bannai-Bannai-Zhu: Relative $t$-designs in binary Hamming association scheme $H(n, 2)$ Des. Codes. Cryptogr. 2017.

For $e \geq 2$ please see the paper:
Bannai-Bannai-Tanaka-Zhu: Tight relative $t$-designs on two shells in hypercubes, and Hahn and Hermite polynomials, Ars Math. Contemp. 2022.
There, the approximate nonexistence of tight relative $2e$-designs on two shells of $H(n, 2)$ is also obtained, under some additional assumptions.

The following papers discuss tight relative $t$-designs on $J(v, k)$.
Bannai-Bannai-Zhu: Tight relative 2-designs on two shells in Johnson association schemes, Discrete Math. 2016,
Bannai-Zhu: Tight $t$-designs on one shell of Johnson association schemes, Europ. J. Comb. 2019.

## $t$-designs on one shell in P-and Q-polynomial association schemes.

Now, we want to see what the Assmus-Mattson theorem means for some known families of P-and Q-polynomial association schemes, in particular for core examples:

1. Johnson association schemes $J(v, d)$.
2. Hamming associationschemes $H(d, q)$.
3. $q$-analogue of Johnson association schemes $J_q(v, d)$ (Grassman association schemes).
4. Association schemes of dual polar spaces of Witt index $d$.
5. Affine association schemes, which consist of
   (i) Associationschemes of bilinear forms $\mathrm{Bil}_{d\times n(q)}$ with $d \leq n$.
   (ii) Association schemes of alternating bilinear forms $\mathrm{Alt}_n(q)$ with $d = [n/2]$.
   (iii) Association schemes of Hermitian forms $\mathrm{Her}_d(q^2)$.
   (iv) Association schemes of quadratic forms $\mathrm{Quad}_n(q)$ with $d = [(n+1)/2]$.

Among the association schemes in this core list, each of them in 1,2,3,4 and 5(i) has an interpretation as the top fiber of a regular semi-lattice (cf. Delsarte (JCT(A),1976), Stanton (Graphs and Comb. 1985).

On the other hand, it seems that the remaining cases, i.e., those in 5(ii)-(iv), are not associated with regular semi-lattice (cf. Terwilliger: Quantum matroids, ASPM, 1996). Nevertheless, Munemasa (Graphs and Comb. 1986), Stanton (Graphs and Comb. 1986) define some graded posets and interpreted $t$-designs geometrically for these cases. We wonder whether there is a good interpretations of relative $t$-designs as well.

Algebraic Definition of $t$-designs on one shell in $J_q(N,a)$.

We adopt the notation used in Dunkl (Monatsh. Math. 1978). Namely, instead of $J_q(v,d)$ we use $J_q(N,a)$. Dunkl studied the harmonic analysis on $J_q(N,a)$ in detail, and determined the set of primitive idempotents of the commutative association scheme $X_i$.

Let $G = GL(N,q)$. We write $N = a + b$, and assume that $a(= d) \leq b$. Let $H$ be the stabilizer of the fixed point $x_0 \in X$ in $G$, which is a maximal parabolic subgroup of $G$. Then the $i$th shell $X_i$ can be written as $X_i = H/H_i$ for some subgroup $H_i$. The decomposition into irreducible representations of the multiplicity-free permutation representation of $H$ on $X_i = H/H_i$ was explicitly described by Dunkl.

It is shown that the primitive idempotents are parametrized as $E_{m,n,r}$ with $0 \leq r \leq i, 0 \leq m \leq (a-i) \wedge (i-r)$, and $0 \leq n \leq (i-r) \wedge (b-i)$. Thus, assuming $t \ll i \ll a$ for simplicity, it follows that the $t$-design in $X_i$ are identified with the harmonic index $T$-design on $X_i$ with $T = \{(m,n,r) \mid 0 < m+n+r \leq t\}$.

**Algebraic Definition of $t$-designs on one shell in dual polar spaces.**

**Very similar results as in the previous cases are also obtained for dual polar spaces. using the work of Stanton, in particular using Theorems 4.10, 5.3 and 5.4 of the following paper.**
**D. Stanton: Three addition theorems for some $q$-Krawtchouk polynomials, (Geom. Dedicata, 1981).**
**(Sorry, I have not checked carefully!)**

**Among the core list of the classical families of P-and Q-polynomial association schemes, those in affine association schemes are remaining. In all cases 5. (i)-(iv) of bilinear forms, alternating forms, Hermitian forms, and quadratic forms association schemes, the shells $X_i$ are not commutative in general. On the other hand, the furthest shell $X_d$ is commutative in all these 4 cases.**

## Algebraic definition of $t$-designs on the furthest shell $X_d$ of affine association schemes.

Here, we consider the case 5.(ii) of the alternating bilinear forms association scheme just for simplicity. So, $X_d = GL(2d,q)/Sp(2d,q)$. The association scheme $X_d$ is commutative, and the set of primitive idempotents of $X_d$ corresponds to the irreducible decomposition of the multiolicity-free permutation representation of $GL(2n,q)$ on $X_d$ which is in turn in bijection with the equivalent irreducible representations of $GL(d,q)$ as described in Bannai-Kawanaka-Song (J. Algebra, 1990). As is well known the irreducible representations of $GL(d,q)$ are parametrized by the functions

$$\mu : \Phi \to \mathcal{P}$$

such that

$$||\mu|| := \sum_{f\in\Phi} (\deg\ f)|\mu(f)| = d,$$

where $\Phi$ is the set of irreducible polynomials $f(x)$ over $\mathbb{F}_q$ distinct from 1 and $x$, and $\mathcal{P}$ is the set of partitions (Young diagrams). Here $|\lambda|$ denotes the weight of a partition $\lambda = (\lambda_1, ..., \lambda_k)$, i.e., $|\lambda| = \lambda_1 + \cdots + \lambda_k$. If we let $E_\mu$ denote the primitive idempotents of $X_d$ corresponding to the irreducible representation of $GL(d,q)$ associated with $\mu$, then it is shown that the harmonic index $T$ for the $t$-designs in $X_d$ consists of those $\mu$ such that the partition $\mu(x-1)$ is of the form $(\mu_1, \ldots, \mu_k)$ with $\mu_1 \geq d-t$, i.e., the size of the first row of the Young diagram is at least $d-t$.

So far we have discussed relative $t$-designs on one shell for the Q-structure for the association schemes in the core list. So, let us consider the same problem for the P-structure.

Definition. Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a P-polynomial association scheme. Fix $x_0 \in X$. We define a subset $Y \subset X_i$ to be a geometric $t$-design on $X_i$, if $Y$ is a relative $t$-design (for the P-structure) on $\mathfrak{X}$ with respect to $x_0$.

Please note that in what follows when we say just a relative $t$-design, we mean a relative $t$-design with the Q-structure. We note that a subset $Y \subset X$ is a geometric $t$-design on $X_i$ if and only if

$$\lambda_t^i(z) = |\{y \in Y \mid z \le y\}|$$

is independent of the choice of $z \in X$, and depends only on $t$ and $i$. Here, for $z \in X_t$ and $y \in X_i, z \le y$ means $t \le i$ and $(z, y) \in R_{i-t}(i.e., f_z(y) = 1$, where $f_z$ in the previous notation).

Note that in $J(v, d)$, the geometric $t$-design on $X_i$ are precisely the harminic index $T$-designs on $X_i$ with $T = \{(k, \ell) \neq (0, 0) \mid 0 \le k, \ell \le t\}$.

## What can we do if shell $X_i$ is not a commutative association scheme ?

The following is rather a speculation. (The idea of considering $W$-design is based on some work of Okuda.)

Let $G$ be a distance-transitive group on $X = G/H$, say. Let $X_i$ be the $i$th shell with respect to $x_0 \in X$.

Let $L_j(X)$ be the $j$-th eigenspace of the Bose-Mesner algebra (onto which $E_i$ is the orthogonal projection).

Let $\pi_i : \mathbb{C}^X \to \mathbb{C}^{X_i}$ be the orthogonal projection which is also $H$-homomorphism. For any $H$-submodule $W$ of $\mathbb{C}^{X_i}$, the concept of a $W$-design $Y \subset X_i$ is defined as follows.

$$\frac{1}{|Y|}\sum_{y\in Y} f(y) = \frac{1}{|X_i|}\sum_{x\in X_i} f(x) \text{ for all } f \in W.$$

The $t$-design on $X_i$, i.e., the relative $t$-designs for Q-structure on the shell $X_i$ with respect to $x_0$, are precisely the $W$-designs on $X_i$, where $H$-submodule $W \subset \mathbb{C}^{X_i}$ is chosen as $W = \pi_i(L_0(X) + L_1(X) + \cdots + L_t(X))$. On the other hand, observe that the map $\rho_{i,t} : \mathbb{C}^{X_t} \to \mathbb{C}^{X_i}$ defined by

$$\rho_{i,t}\phi_z = \sum_{x\in X_i, (x,z)\in R_i} \phi_x \text{ for all } z \in X_t,$$

is an $H$-homomorphism. The geometric $t$-design on $X_i$, i.e., the relative $t$-design for the P-structure on $X_i$ with respect to $x_0$, are the $W$-design on $X_i$, where in this case $W = \rho_{i,t}\mathbb{C}^{X_i}$.

If $X_i$ is commutative, if the permutation representation of $H$ on $X_i$ is multiplicity-free, then any $H$-submodule $W \subset \mathbb{C}^{X_i}$ is also multiplicity-free. In particular, this implies that the $t$-design as well as geometric $t$-designs on $X_i$ are always harmonic index $T$-designs on $X$ for some appropriate $T$ of an index set of the primitive idempotents of $X_i$. However, the situation could be very delicate if $X_i$ is not commutative. Still, I think it is worth studying the structures of these $H$-submodules $W$.