G2C2 Lecture No. 4. Euclidean t-designs

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Basic References

[1] Bannai-Bannai-Ito-Tanaka: Algebraic Combinatorics, Chapter 5, De Gruyter, 2021.

[2] Bannai-Bannai-Tanaka-Zhu: DesignTheory from the viewpoint of algebraic combinatorics, Graphs and Comb. 31(2017), 1-41.

We will give more references later.

The concept of Euclideans t-designs

Now, we want to define Euclidean t-designs, as a two step generalization of spherical t-designs.

Today, we restrict our discussion to generalizations of spherical t-designs. We will not discuss generalizations of combinatorial designs here, although a similar theory has already been developed for the combinatorial t-designs.

First step generalization (weighted spherical t-design, or cubature formula on the sphere).

- Let $X \subset S^{n-1}$, and let $w : X \longrightarrow \mathbb{R}_{>0}$. Then the pair (X, w) is called a weighted spherical t-design, if

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x) = \sum_{x\in X} w(x)f(x),$$

for any polynomial $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most t.

The second step generalization (allow several concentric spheres).

- A weighted subset (X, w) on a union
$S = S_1^{n-1}(r_1) \cup S_2^{n-1}(r_2) \cup \cdots \cup S_p^{n-1}(r_p)$
of p concentric spheres is called a Euclidean t-design on S if

$$\sum_{i=1}^{p} \frac{W(X_i)}{|S_i^{n-1}(r_i)|} \int_{S_i^{n-1}(r_i)} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x)$$

holds for any polynomial $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most t.
Here, $X_i = X \cap S_i^{n-1}(r_i)$ and $W(X_i) = \sum_{x \in X_i} w(x)$ for $1 \leq i \leq p$.

So, we have the following definition of Euclidean t-designs (on a union of p concentric spheres).

Definition. (Neumaier-Seidel, 1988). (X, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{W(X_i)}{|S_i^{n-1}(r_i)|} \int_{S_i^{n-1}(r_i)} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x)$$

for any polynomial $f(x)$ of degree at most t,
where $X_i = X \cap S_i^{n-1}(r_i)$ and $W(X_i) = \sum_{x \in X_i} w(x)$.

The definition of Euclidean *t*-design was first introduced by Neumaier-Seidel in 1988. There were similar concepts in statistics as rotatable designs and also in numerical analysis as cubature formulas on radially symmetric measures.

I am personally indebted to J. J. Seidel, as he wanted me to study tight Euclidean *t*-designs. Jointly with Etsuko Bannai, we started a systematic study of tight Euclidean *t*-designs, in particular in [BB, On Euclidean tight 4-designs, JMSJ](2006).

Now, we will give a survey of the study of Euclidean *t*-designs.

General Theory of Euclidean designs

Let $\mathcal{P}_e(\mathbb{R}^n)$ be the space of polynomials of degree at most e on $\mathbb{R}^n$. $\mathcal{P}_e(S^{n-1})$ is the space obtained by restricting it to the set $S = S^{n-1}(r_1) \cup S^{n-1}(r_2) \cup \cdots \cup S^{n-1}(r_p)$. Let X be a finite set on S. (Here we assume that the r_i are all distinct and positive for simplicity.) Then

$$\dim \mathcal{P}_e(S) = \sum_{i=0}^{2p-1} \binom{n-1+e-i}{e-i}.$$

Note that $|X| \geq \dim P_e(S)$ for Euclidean $t(=2e)$ design on p concentric spheres.

Let $\mathcal{P}_e^*(\mathbb{R}^n)$ be the subspace of $\mathcal{P}_e(\mathbb{R}^n)$ all of their terms have the same parity as e. Then

$$\dim \mathcal{P}_e^*(S) = \sum_{i=0}^{p-1} \binom{n-1+e-2i}{e-2i}.$$

Note that $|X| \geq 2 \dim \mathcal{P}_e^*(S)$ for Euclidean $t(=2e+1)$ design on p concentric spheres.

We call X to be a tight Euclidean t-design on a union S of p concentric spheres, if $|X| = \dim \mathcal{P}_e(S)$ for $t = 2e$; and $|X| = 2 \dim \mathcal{P}_e^*(S)$ for $t = 2e + 1$.

The following conditions are equivalent (Neumaier-Seidel(1988))

(1) X **is a Euclidean** t**-design with weight function** w**.**

(2) $\sum_{x \in X} w(x) f(x) = 0$**, for any polynomial** $f \in \|x\|^{2j}\mathrm{Harm}_{\ell}(\mathbb{R}^n)$ **with** $1 \leq \ell \leq t$ **and** $0 \leq j \leq [\frac{t-\ell}{2}]$**.**

The condition (2) is furthermore modified in a form suitable for further applications. I will not write down these modified forms, as it is very complicated. We may call them "Fundamental Equations".(See (2.3) and (2.4) in [BB, On Euclidean tight 4-designs, JMSJ (2006)] for $t = 2e$ **and see (3.1) and (3.2) in [Etsuko B, On antipodal Euclidean tight** $(2e+1)$**-designs, (2009)] for** $t = 2e+1$**.**

From these "Fundamental Equations", we can obtain the following facts.

<u>Theorem</u>.

(i) Let (X, w) be a tight Euclidean t-design on p concentric spheres. Then the weight function $w(x)$ is constant on each X_i.

(ii) Let (X, w) be a tight Euclidean t-designs on 2 concentric spheres (namely, $p = 2$). Then X has the structure of a coherent configuration, with respect to the inner products as relations.

Moreover, if $t = 2e$, then this coherent configuration is of type $\begin{bmatrix} e & e \\ & e+1 \end{bmatrix}$ or $\begin{bmatrix} e+1 & e \\ & e+1 \end{bmatrix}$; and if $t = 2e + 1$ then it is of type $\begin{bmatrix} e+1 & e \\ & e+2 \end{bmatrix}$ or $\begin{bmatrix} e+2 & e \\ & e+2 \end{bmatrix}$.

(Notation will be explained in the next page.)

(iii) Moreover, if $p = 2$, then X_1 and X_2 are spherical $(t-2)$-designs on S_1 and S_2, respectively.

It is also known (by Möller) that if (X, w) is a Euclidean tight $t = (2e + 1)$-design (and for any p), then X is antipodal. (Namely, $x \in X$ implies $-x \in X$.)

Coherent configurations.

Here we should note that the coherent configuration is a purely combinatorial object generalizing the association scheme. Namely $(X, \{R_i\}(i \in I))$, a pair of a finite set X and a set of relations $R_i (i \in I)$ on X, is called a coherent configuration, if the following conditions (1) to (4) are satisfied:

(1) $\{R_i(i \in I)\}$ gives a partition of $X \times X$.
(2) There is a subset $S \subset I$, such that $\{(x, x) \mid x \in X\} = \bigcup_{i \in S} R_i$,
(3) For each $i \in I$, there exist $i' \in I$ such that ${}^t R_i = R_{i'}$ where
${}^t R_i = \{(y, x) \mid (x, y) \in R_i\}$.
(4) For any fixed $i, j, k \in I$, the number
$|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p^k_{i,j}$ is constant whenever $(x, y) \in R_k$ holds.

Note that the association scheme is the special case of the coherent configuration with $|S| = 1$. Also note that, just like the association schemes are useful for the study of spherical designs, coherent configurations are useful for the study of Euclidean designs.

The type of the coherent configuration is the $|S| \times |S|$-matrix whose (i, j)-entry is the number of relations R_k in $X_i \times X_j$, where
$X_i = \{x \in X \mid (x, x) \in R_i\}$, for $i \in S$.

Tight Euclidean 2-and 3-designs.

Tight Euclidean 2-designs are characterized as one inner product sets with negative inner product value, although the one inner product sets are not yet completely understood. (See [B-B-Suprijanto, 2010].) On the other hand, tight Euclidean 3-designs are completely determined. See [Etsuko B, On antipodal Euclidean tight $(2e+1)$-designs (2009)]. Namely, up to an orthogonal transformation, we have $X = X_1 \cup X_2 \cup \cdots \cup X_p$ with

$$X_i = \{\pm r_i e_j \mid 1 + \sum_{\ell=1}^{i-1} N_\ell \leq j \leq \sum_{\ell=1}^{i} N_\ell\}.$$

Here, $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$, $2N_i = |X_i|$ and
$w(x) = \frac{1}{nr_i^2}$ for $x \in X_i$ and for $i = 1, 2, \ldots, p$.

So, tight Euclidean t-designs with $t = 2$ and 3 were classified for any p. However, it is not so easy to try to do the classification for $t \geq 4$, for general p.
So, we will try to classify those, first for $p = 2$.

Review of the classification of tight Euclidean t-designs on two concentric spheres.

First, we remark that tight Euclidean t-design in $\mathbb{R}^2$ are classified by Bajnok [Bajnok, 2006], under the assumption that $p \leq [\frac{t}{4}] + 1$. See [Bajnok, On Euclidean designs, Advances in Geometry, 2006], (Cf. also [B-B-Hirao-Sawa, Cubature formulas in numerical analysis and Euclidean tight designs, Europ. J. Comb. 2010].) (Also in the area of numerical analysis: these examples were considered by Verlinden-Cools (1992), Cools-Schmid (1993).)
(The classification of tight Euclidean t-design in $\mathbb{R}^2$ is still wide open for bigger p.)

There is a work of Etsuko Bannai that gives the complete parameters of tight Euclidean 5 designs of $\mathbb{R}^2$. (There are plenty of freedoms on the parameters.)

Etsuko Bannai: Classification of tight Euclidean 5-designs in $\mathbb{R}^2$, RIMS Proceedings No. 2253, 2023.

Tight Euclidean t-designs for $t = 2e + 1$ and $p = 2$.

Tight Euclidean t-designs on two concentric spheres (namely with $p = 2$) was completely solved for $t = 5, 7$ and 9, in the following papers, respectively.
(1) Etsuko Bannai: On antipodal Euclidean tight $(2e + 1)$-designs, J. Alg. Comb., 2006], for $t = 5$.
(2) Bannai-Bannai [Beijing, Higher Edu. Press, 2009], for $t = 7$.
(3) Bannai-Bannai [JMSJ, 2010] for $t = 9$.

Moreover, it is proved that

(4) For each fixed $t \geq 13$ (odd), there are only finitely many n for which there exists a tight Euclidean t-design in $\mathbb{R}^n$ with $p = 2$.
Bannai-Bannai [Moscow J. Comb. Number Theory, 2014].

The finiteness of n for $t = 11$, is not yet obtains, although there are no nontrivial example known to exist.

Tight Euclidean 5-designs for $p = 2$ was classified by [Etsuko B, On antipodal Euclidean tight $(2e + 1)$- designs, J. Alg. Comb., 2006]:

Theorem. Let X be an (antipodal) tight Euclidean 5-design in $\mathbb{R}^n$ supported by 2 concentrated spheres. Then X is similar to one of the following: Here, X_1 is on the unit sphere and X_2 is on the sphere of radius $r \neq 1$.

- $n = 2$, $X = X_1 \cup X_2$, $X_1 = \{(\pm 1, 0), (0, \pm 1)\}$, $X_2 = \{(\pm \frac{r}{\sqrt{2}}, \pm \frac{r}{\sqrt{2}})\}$, $r \neq 1$, $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{1}{r^4}$ for $x \in X_2$.

- $n = 3$, $X_1 = \{\pm e_i \mid i = 1, 2, 3\}$, $X_2 = \{\frac{r}{\sqrt{3}}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i \in \{1, -1\},\ i = 1, 2, 3\}$, $r \neq 1$, $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{9}{8r^4}$ for $x \in X_2$.

- $n = 5$, $X = X_1 \cup X_2 \subset V = \{(x_1, x_2, \ldots, x_6) \in \mathbb{R}^6 \mid \sum_{i=1}^{6} x_i = 0\} \cong \mathbb{R}^5$, $X_1 = \{\pm u_i \mid 1 \leq i \leq 6\}$, $X_2 = \left\{\frac{r}{\sqrt{6}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \mid \varepsilon_i \in \{1,\ -1\},\ |\{i \mid \varepsilon_i = 1\}| = 3\right\}$, $r \neq 1$ where $u_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,6}) \in V,\ 1 \leq i \leq 6,\ u_{i,i} = -\frac{5}{\sqrt{30}},\ u_{i,j} = \frac{1}{\sqrt{30}}$ for $j \neq i$, $1 \leq j \leq 6$. $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{27}{25r^4}$ for $x \in X_2$.

- $n = 6$, $X_1 = \{\pm e_i \mid 1 \leq i \leq 6\}$, where $\{e_1, e_2, \ldots, e_6\}$ is the canonical basis of $\mathbb{R}^6$. $X_2 = \left\{\frac{r}{\sqrt{6}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \;\middle|\; \begin{array}{l} \varepsilon_i \in \{1,\ -1\}, \\ |\{i \mid \varepsilon_i = 1\}| \equiv 0 \pmod 2 \end{array}\right\}$, where $r \neq 1$, $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{9}{8r^4}$ for $x \in X_2$.

The following result is in [BB, "Spherical designs and Euclidean designs" Higher Education Press (2009)].

<u>Theorem</u> A Euclidean tight 7-design (X, w) supported by a union of two concentric spheres in $\mathbb{R}^n$ $(0 \notin X)$ is similar to one of the following: X_1 is on the unit sphere and X_2 is on the sphere of radius $r \neq 1$.

- $n = 2$: $|X| = 12$, $|X_1| = 6$, $|X_2| = 6$, $w(x) = 1$, for $x \in X_1$, $w(x) = \frac{1}{r^6}$ for $x \in X_2$, X_1 and X_2 are 3-distance sets. X_1 and X_2 are spherical tight 5-designs.
- $n = 4$: $|X| = 48$, $|X_1| = |X_2| = 24$, $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{1}{r^6}$ for $x \in X_2$, $r \neq 1$. X_1 and X_2 are 4-distance sets.
- $n = 7$: $|X| = 182$, $|X_1| = 56$, $|X_2| = 126$, $w(x) = 1$ for $x \in X_1$, $w(x) = \frac{32}{27}\frac{1}{r^6}$ for $x \in X_2$, $r \neq 1$. X_1 is a spherical tight 5-design which is a 3-distance set, X_2 is a 4-distance set.

<u>Theorem</u> [BB, JMSJ, 2009]: Let (X, w) be a Euclidean tight 9-design supported by a union of two concentric spheres in $\mathbb{R}^n$ $(0 \notin X)$. Then we must have $n = 2$ and $|X_1| = |X_2| = 8$. X_1 and X_2 are regular 8-gons. $w(x) = 1$ for $x \in X_1$ and $w(x) = \frac{1}{r^8}$ $(r \neq 1)$ for $x \in X_2$

Tight Euclidean t-designs for $t = 2e$ and $p = 2$

One example of tight Euclidean 6-design with $p = 2$ ($n = 22, |X_1| = 275$, $|X_2| = 2025$) was found by B-B-Shigezumi, A new tight Euclidean 6-design Ann. Comb. (2012). Besides this work, tight Euclidean $t = 2e$ designs have not been much studied, in particular for $e \geq 3$. So, here we mostly consider

Tight Euclidean 4-designs with $p = 2$

Let (X, w) be a tight Euclidean 4-design on two concentric spheres $S = S^{n-1}(r_1) \cup S^{n-1}(r_2)$. Then $|X| = \frac{(n+2)(n+1)}{2}$.

It was shown that X has the structure of coherent configuration $\begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$ or $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$. Moreover we get the following assertions:

(i) X_1 and X_2 are spherical 2-designs on $S^{n-1}(r_1)$ and $S^{n-1}(r_2)$ respectively.

(ii) w is constant on each X_i $(i = 1,\ 2)$.

(iii) Without loss of generality, we may assume that
$w(x) \equiv w_1 = 1$ for $x \in X_1$, and $w(x) \equiv w_2$ for $x \in X_2$,
and $r_1 = 1$, $r_2 = r(\neq 1)$

(iv) Type $\begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$ coherent configuration means X_1 is a tight spherical 2-design (a regular simplex) on $S^{n-1}(r_1) \subset \mathbb{R}^n$. Type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$ coherent configuration means X_1 and X_2 are SRGs on $S^{n-1}(r_1)$ and $S^{n-1}(r_2)$ respectively.

The following result was proved by [Etsuko B, New Examples of Euclidean tight 4-designs, Europ. J. Comb., 2009].

<u>Theorem</u>. [Etsuko B] (2009).
Let (X, w) be a Euclidean tight 4-design on a union $S_1(r_1) \cup S_2(r_2)$ of two concentric spheres of radius r_1 and r_2. Let $X = X_1 \cup X_2$. In the following $A(X_i, X_j) = \{x \cdot y \mid x \in X_i, y \in X_j, x \neq y\}$, where $x \cdot y$ is the canonical inner product of x and y. If $|X_1| = n + 1$, then (X, w) is similar to one of the following.

(1) $n = 2$, $|X_1| = |X_2| = 3$, $r_1 = 1$, $r_2 = r \neq 1$, $w_1 = 1$, $w_2 = \frac{1}{r^3}$,
$A(X_1) = \{-\frac{1}{2}\}$, $A(X_2) = \{-\frac{1}{2}r^2\}$, $A(X_1, X_2) = \{\frac{1}{2}r, -r\}$.

(2) $n = 4$, $|X_1| = 5$, $|X_2| = 10$, $r_1 = 1$, $r_2 = \frac{1}{\sqrt{6}}$, $w_1 = 1$, $w_2 = 27$,
$A(X_1) = \{-\frac{1}{4}\}$, $A(X_2) = \{\frac{1}{36}, -\frac{1}{9}\}$, $A(X_1, X_2) = \{\frac{1}{6}, -\frac{1}{4}\}$.

(3) $n = 5$, $|X_1| = 6$, $|X_2| = 15$, $r_1 = 1$, $r_2 = \sqrt{\frac{8}{5}}$, $w_1 = 1$, $w_2 = \frac{1}{2}$,
$A(X_1) = \{-\frac{1}{5}\}$, $A(X_2) = \{\frac{2}{5}, -\frac{4}{5}\}$, $A(X_1, X_2) = \{\frac{2}{5}, -\frac{4}{5}\}$,

(4) $n = 6$, $|X_1| = 7$, $|X_2| = 21$, $r_1 = 1$, $r_2 = \sqrt{15}$, $w_1 = 1$, $w_2 = \frac{1}{81}$,
$A(X_1) = \{-\frac{1}{6}\}$, $A(X_2) = \{\frac{9}{2}, -6\}$, $A(X_1, X_2) = \{1, -\frac{5}{2}\}$.

(5) $n = 22$, $|X_1| = 23$, $|X_2| = 253$, $r_1 = 1$, $r_2 = \sqrt{\frac{126}{11}}$,
$w_1 = 1$, $w_2 = \frac{1}{81}$, $A(X_1) = \{-\frac{1}{22}\}$, $A(X_2) = \{\frac{45}{22}, -\frac{117}{44}\}$,
$A(X_1, X_2) = \{\frac{21}{44}, -\frac{12}{11}\}$.

In above, for $n = 4,\ 5,\ 6$, X_2 has the structure of the Johnson association scheme $J(n+1,2)$. For $n = 22$, X_2 has the structure of tight 4-$(23,7,1)$ design in the Johnson scheme $J(23,7)$.

If $|X_1| = n+2$, then $n = 4$ and (X,w) is similar to the Euclidean tight 4-design with the parameters given below. Moreover X_2 has the structure of the Hamming scheme $H(2,3)$, that is, trivial tight 4-design of the Hamming scheme.

$|X_1| = 6$, $|X_2| = 9$, $r_1 = 1$, $r_2 = \sqrt{2}$, $w_1 = 1$, $w_2 = \frac{1}{3}$,
$A(X_1) = \{0, -\frac{1}{2}\}$, $A(X_2) = \{\frac{1}{2}, -1\}$, $A(X_1, X_2) = \{\frac{1}{2}, -1\}$.

If $|X_1| \geq n+3$ and $2 \leq n \leq 77$. Then $n = 22$, $|X_1| = 33$, and (X,w) is similar to the Euclidean tight 4-design with the parameters given below. Moreover X_2 has the structure of tight 4-design in the Hamming scheme $H(11.3)$.

$|X_1| = 33$, $|X_2| = 243$, $r_1 = 1$, $r_2 = \sqrt{11}$, $w_1 = 1$, $w_2 = \frac{1}{81}$,
$A(X_1) = \{0, -\frac{1}{2}\}$, $A(X_2) = \{2, -\frac{5}{2}\}$, $A(X_1, X_2) = \{\frac{1}{2}, -1\}$.

Now, I would like to explain some new ongoing developments for the study of this case $|X_1| \geq n+3$.

So, in what follows we may assume without loss of generality that

$$n+3 \leq |X_1| \leq |X_2| \leq \binom{n+2}{2} - (n+3),$$

and the c.c. is of type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$, since the cases $|X_1| = n+1$ and $n+2$ were already solved in [Etsuko B](2009).

Now we have a new:

Theorem. Let (X, w) be a tight Euclidean 4-design on two concentric spheres in $\mathbb{R}^n$. Then $n = (2m+1)^2 - 3$ for some positive integer m. (Here, $|X_1| \geq n+3$ is assumed as mentioned above.)
(This was conjectured in [Etsuko B,2009] and [BB, 2010].)

Now, we want to discuss the context of our paper: [BB, Contemporary Math.](2010): Euclidean designs and coherent configurations.
and some new developments.

The basic idea of the proof in [Etsuko B, 2009] and [BB, 2010] is as follows. The Fundamental Relations were very basic. Using that we can determine the (normalized) inner product sets
$A(X_1, X_1) = \{\alpha_1, \alpha_2\}, A(X_2, X_2) = \{\beta_1, \beta_2\}$ and $A(X_1, X_2) = \{\gamma_1, \gamma_2\}$ to be expressed only using the parameters n and N_1, say.

Then using the fact that X_1 and X_2 are both 2-distance sets, using the Larman-Rogers-Seidel's theorem (1977), we expect that $(\frac{2-\alpha_1-\alpha_2}{\alpha_1-\alpha_2})^2$ and $(\frac{2-\beta_1-\beta_2}{\beta_1-\beta_2})^2$ to be the square of some odd integers.

In fact, just very recently (in 2021), we succeeded in proving that

$$\left(\frac{2-\alpha_1-\alpha_2}{\alpha_1-\alpha_2}\right)^2 = \left(\frac{2-\beta_1-\beta_2}{\beta_1-\beta_2}\right)^2 = n+3.$$

And this gives the Theorem.

Theorem.(Theorem 1.8 in [BB, Contemporary Math](2010).)

The following is a family of feasible parameters for tight Euclidean 4-design in $\mathbb{R}^n$. That is, the condition that the intersection numbers $p_{i,j}^k$ of associated coherent configuration are all nonnegative integers is satisfied.

$n = (6k-3)^2 - 3$, with any positive integer k,

$|X_1| = (6k^2-6k+1)(36k^2-36k+7), \quad |X_2| = 3(36k^2-36k+7)(2k-1)^2,$

$A(X_1) = \left\{ \frac{18k^2-27k+8}{6(9k^2-9k+1)(2k-1)}, -\frac{18k^2-9k-1}{6(9k^2-9k+1)(2k-1)} \right\},$

$A(X_2) = \left\{ \frac{36k^3-54k^2+25k-4}{2(6k^2-6k+1)(18k^2-18k+5)}, -\frac{36k^3-54k^2+25k-4}{2(6k^2-6k+1)(18k^2-18k+5)} \right\},$

$A(X_1, X_2) = \left\{ \sqrt{\frac{36k^2-36k+4}{(36k^2-36k+6)(36k^2-36k+10)}}, -\sqrt{\frac{36k^2-36k+10}{(36k^2-36k+6)(36k^2-36k+4)}} \right\},$

$r_1 = 1, \ r_2 = \sqrt{\frac{3(18k^2-18k+5)(6k^2-6k+1)}{9k^2-9k+1}},$

$w(x) = 1$ for $x \in X_1$ and $w(x) = \frac{1}{81(2k-1)^4}$ for $x \in X_2$.

Our original observation was that the feasible parameter set in Theorem 1.8 (in [BB, 2010) may be the only possible parameters of such tight Euclidean 4-designs with $p = 2$, except for some already known small solutions, and we confirmed up to certain n.

Moreover, we could prove the following holds. Let (X, w) be a tight Euclidean 4-design on two concentric spheres in $\mathbb{R}^n$. Then it must be one of the following possibility holds. (i) It must be one of those listed in this slide. or (ii) the parameters (of the coherent configuration) must be those listed in Theorem 1.8 of [BB, Contemporary Math](2010), or (iii) the following diophantine equation must have integer solutions.

$$\begin{aligned}
&-16(n+1)^2(n^2+3n-2x+2)y^4+32x(n+1)^2(n^2+3n-2x+2)y^3\\
&-8nx^2(3n+7)(n^2+3n-2x+2)y^2\\
&+8x^3(n^2+3n-2)(n^2+3n-2x+2)y\\
&-x^3(x-1)(n+2)^2(n^2+3n-2x)=0.
\end{aligned}$$

As soon as we showed this to Ziqing Xiang, he found some exceptional solutions for this diophantine equations, and for $n \leq 10^6$, the following are the only solutions (Ziqing Xiang).

Here $(n, x = N_2, y = m_2)$ with $\frac{(n+2)(n+1)}{4} \leq x \leq \frac{(n+2)(n+1)}{2} - (n+3)$, and we assume $22 \leq m \leq 1000$.

$(n, x, y) =$:
22, 243, 162
838, 250563, 127738(*)
2806, 2861937, 1448811
2806, 3924375, 2130375(*)
235222, 22895695251, 11476301265(*)
235222, 26705685501, 13421360205
248998, 22520234375, 11275031250
373318, 55746874041, 27921744537
373318, 68089612943, 34202664925
3606198, 5623031516635, 2813618431520
3606198, 6138212423005, 3072520122750(*)

(It is NOT known whether this diophantine equation has only finitely many exceptional solutions or not.)
By considering the parameters of the corresponding coherent configuration, we can get contradictions (hence the non-existence of) for the first 4 cases indicated by (*).

So, this is the most updated situation of the classification problem of tight Euclidean 4-designs with $p = 2$. The situation is still very delicate. We would be happy, if any of you challenge this situation.

Etsuko and I have been visiting Taipei (National Taiwan University) for December 2020-January 2022. We have been working with some researchers in Taiwan, including Ziqing Xiang, Wei-Hsuan Yu, Chin-Yen Lee, and several students. Here is one new result. [B, B, Lee, Xiang, Yu, arXiv:2211.02331]

- **Let X be a type $\begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$ coherent configuration embedded in $\mathbb{R}^n$. Suppose that X is a 2-distance set in $\mathbb{R}^n$. Then $n = 8$ and X must be the 45 point set of Lisoněk, JCT(A), 1997.**

Thank you very much