G2C2 Lecture No. 3. Tight t-designs

Eiichi Bannai

August 13, 2024

Basic References

[1] Delsarte: An algebraic approach to the association schemes of the coding theory, Thesis, Philips Res. Repts. Suppl. 10, 1973.

[2] Bannai-Ito: Algebraic Combinatorics I: Association Schemes, Benjamin-Cummings, 1984.

[3] Bannai-Bannai-Ito-Tanaka: Algebraic Combinatorics, Chapter 2, De Gruyter, 2021.

We will give more references later.

We first consider Gegenbauer polynomials.

Let $w(x)$ be the real function on the interval $[-1, 1]$ such that $w(x) = (\mathrm{const}) \times (1 - x^2)^{\frac{n-3}{2}}$ with

$$\int_{-1}^{1} w(x)dx = 1.$$

The Gegenbauer polynomials $Q_i(x)$ are the real coefficients polynomials satisfying:

$$\int_{-1}^{1} Q_i(x)Q_j(x)w(x)dx = \delta_{i,j}\alpha_i$$

where α_i are positive real numbers. If we normalize so that

$$Q_i(1) = \dim(\mathrm{Harm}_i(\mathbb{R}^n)) = \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2},$$

then they are determined uniquely. Actually, they are expressed as follows. (Note that there are some other normalizations, then the form of the polynomials is slightly different with those used here.)

We have as follows.

$Q_0(x) = 1,$
$Q_1(x) = nt,$
$2Q_2(x) = n(n+2)(t^2 - \frac{1}{n}),$
$6Q_3(x) = n(n+2)(n+4)(t^3 - \frac{3t}{n+2}),$
$24Q_4(x) = n(n+2)(n+4)(n+6)\left(t^4 - \frac{6t^2}{n+4} + \frac{3}{(n+2)(n+4)}\right),$
$120Q_5(x) = n(n+2)(n+4)(n+6)(n+8)\left(t^5 - \frac{10t^3}{n+6} + \frac{15t}{(n+4)(n+6)}\right).$

General formula is as follows.

$$Q_\ell(t) = Q_{\ell,n}(t) = \sum_{j=0, i-j \equiv 0 (mod\ 2)}^{\ell} c_{\ell-j} t^j (1-t^2)^{\frac{\ell-j}{2}},$$

where $c_0 = Q_\ell(1) (= h_\ell)$ and each c_i, $(i = 2, 4, ..., \leq \ell)$ are defined by $c_{\ell-j}(\ell-j)(\ell-j+n-3) + (j+1)(j+2)c_{\ell-j-2} = 0.$

The following three term recurrence relations hold.

$$\frac{k_\ell}{k_{\ell+1}} Q_{\ell+1}(t) = tQ_\ell(t) - (1 - \frac{k_{\ell-2}}{k_{\ell-1}}) Q_{\ell-1}(t),$$

where k_ℓ is the coefficient of the highest degree ℓ of the polynomial $Q_\ell(t)$ and satisfies $\frac{k_\ell}{k_{\ell+1}} = \frac{\ell+1}{n+2\ell}$.

Note that the space of all the polynomials $P(S^{n-1})$ on the sphere S^{n-1} is decomposed as:

$$P(S^{n-1})(= Harm_0(S^{n-1}) \perp Harm_1(S^{n-1}) \perp Harm_2(S^{n-1}) \perp \cdots\cdots.$$

The following "Addition Theorem" is known.

- For any $x,\ y \in S^{n-1}$ and any non-negative integer ℓ, and for any orthonormal basis $\phi_{\ell,1}, \phi_{\ell,2}, ..., \phi_{\ell,h_\ell}$ (where $h_\ell = \dim Harm_\ell(S^{n-1}) = \dim Harm_\ell(\mathbb{R}^n)$), we have

$$\sum_{i=1}^{h_\ell} \phi_{\ell,i}(x)\phi_{\ell,i}(y) = Q_\ell(x \cdot y)$$

where $x \cdot y$ is the usual inner product in $\mathbb{R}^n$.

Recall that any real coefficient polynomial $F(x)$ is expressed as

$$F(x) = \sum_{k=0}^{\infty} f_k Q_k(x) \quad \text{(finite sum)} \ .$$

(called the Gegenbauer expansion).

Theorem. Suppose the Gegenbauer polynomial expression of a real coefficient polynomial $F(x)$ is given

$$F(x) = \sum_{k=0}^{\infty} f_k Q_k(x) \quad \text{(finite sum)}$$

and the following conditions are satisfied:

$$F(a) \geq 0 \text{ for any } a \in [-1,1], F(1) > 0, f_0 > 0, \text{ and } f_k \leq 0 \text{ for any } k \geq t+1.$$

Then, for a spherical t-design X on S^{n-1}, the following condition holds:

$$|X| \geq \frac{F(1)}{f_0}.$$

Proof.

We use the following notation. $A(X) = \{x \cdot y \mid x, y \in X, x \neq y\}$ and $A'(X) = A(X) \cup \{1\}$. Also, for each $\alpha \in A'(X)$ the matrix D_α whose rows and columns are indexed by the elements of X. Namely, $D_\alpha(x,y) = 1$, if $x \cdot y = \alpha$, and $D_\alpha(x,y) = 0$, if $x \cdot y \neq \alpha$. For each α, define

$$d_\alpha = \sum_{(x,y)\in X\times X} D_\alpha(x,y).$$

(Note that $D_1 = I$ and $d_1 = |X|$.)

Recall that the characteristic matrix H_k for $X \subset S^{n-1}$ is the $|X| \times h_k$-matrix where the (x,i)-entry is $\phi_{k,i}(x)$ where $\{\phi_{k,1}(x), \phi_{k,2}(x), \ldots, \phi_{k,h_k}(x)\}$ is a fixed orthonormal basis of the space $Harm_k(\mathbb{R}^n)$. Note that X is a spherical t-design if and only if $||{}^tH_kH_0|| = 0$. i.e., equivalent to $\sum_{x\in X} \phi_{k,i}(x) = 0$ for all i with $1 \leq i \leq h_k$. Now, we have

$$||{}^tH_kH_0||^2 = \sum_{i=1}^{h_k}(({}^tH_kH_0)(i))^2 = \sum_{i=1}^{h_k}(\sum_{x\in X} H_k(x,i))^2 = \sum_{i=1}^{h_k}(\sum_{x\in X} \phi_{k,i}(x) \sum_{y\in X} \phi_{k,i}(y))$$

$$= \sum_{(x,y)\in X\times X} \sum_{i=1}^{h_k} \phi_{k,i}(x)\phi_{k,i}(y) = \sum_{(x,y)\in X\times X} Q_k(x\cdot y) = \sum_{\alpha\in A'(X)} d_\alpha Q_k(\alpha).$$

Therefore, we obtain

$$\sum_{k=0}^{(\infty)} f_k ||{}^t H_k H_0||^2 = \sum_{k=0}^{(\infty)} f_k \sum_{\alpha\in A'(X)} d_\alpha Q_k(\alpha) = \sum_{\alpha\in A'(X)} d_\alpha F_(\alpha)$$

$$= |X|F(1) + \sum_{\alpha\in A(X)} d_\alpha F(\alpha).$$

Moreover, since X is a t-design, we have

$$f_0||{}^t H_0 H_0||^2 - |X|F(1) = \sum_{\alpha\in A(X)} d_\alpha F(\alpha) - \sum_{k=t+1}^{(\infty)} f_k ||{}^t H_k H_0||^2 \qquad .....(2)$$

Thus we get $f_0|X|^2 - |X|F(1) \geq 0$ by the assumption. So, we get $|X| \geq \frac{F(1)}{f_0}$. Also, if the equality $|X| = \frac{F(1)}{f_0}$ holds, then the right hand side of the above inequality (2) must be equal to 0. Since $d_\alpha > 0$, for any $\alpha \in A(X)$ and $f_k \leq 0$ for any $k \geq t+1$, we have $F(\alpha) = 0$ for any $a \in A(X)$ and $f_k||{}^t H_k H_0||^2 = 0$ for any $k \geq t+1$.

Theorem(Fisher type inequality for spherical $2e$-designs).
Let X be a spherical $2e$-design on S^{n-1}. Then the following inequality holds:

$$|X| \geq \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1} = R_e(1),$$

where $R_e(x) = Q_0(x) + Q_1(x) + \cdots + Q_e(x)$. Moreover, the equality holds, if and only if the set $A(X)$ is equal to the set of zeros of $R_e(x)$.

Proof. Use $F(x) = R_e(x)^2$. Then we can show that $F(1) = R_e(1)^2$ and $f_0 = R_e(1)$ (need some argument). So, we get

$$|X| \geq F(1)/f_0 \geq R_e(1) = \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}.$$

QED

Theorem(Fisher type inequality for spherical $(2e+1)$-designs.
Let X be a spherical $(2e+1)$-design on S^{n-1}. Then we have the following inequality:

$$|X| \geq 2\binom{n-1+e}{e} = 2C_e(1),$$

where $C_e(x) = Q_e(x) + Q_{e-2}(x) + \cdots + Q_{e-2[\frac{e}{2}]}(x)$. Moreover, equality holds if and only if $A(X)$ consists of -1 and the zeros of $C_e(x)$.
(This time, take $F(x) = (x+1)(C_e(x))^2$.)

As we have discussed yesterday, those X with equality in one of the above two theorems are called tight t-designs ($t = 2e$ or $t = 2e+1$) and they are almost classified, exactly speaking exept for $t = 4, 5, 7$. These Fisher type bound are closely related to some bound for s-distance sets in S^{n-1}. Here, I will mention some general results, without giving the details. Please check the references for further details.

Theorem.
Let X be a finite sbset of S^{n-1}.
(i) If X is an s-distance set and t-design, we have $t \leq 2s$ in general.
(ii) If X is an s-distance set, then $|X| \leq R_s(1) = \binom{n-1+s}{s} + \binom{n-1+s-1}{s-1}$. If the equality attains here, then X is a $2s$-design, hence a tight spherical $2s$-design. (We have already mentioned that if X is a tight $2s$-design, then it is an s-distance set.)

More on t-designs on Q-polynomial association schemes.

Yesterday, we defined the concept of t-designs on Q-polynomial association schemes, following Delsarte.

We first remark that the Hamming association scheme $H(d, q)$ and Johnson association scheme $J(v, d)$ are examples of P-polynomial association schemes and also of Q-polynomial association schemes. (We need some further discussions to show why they are Q-polynomial association schemes.)

The following definition of combinatorial t-design t-(v, k, λ) design is well known.

<u>Definition (Combinatorial t-design)</u>.

Let V be a set of v element. Let k be a number $1 \le k \le v$ and $\binom{V}{k}$ be the set of all the k-element subsets of the set V. Let $\mathbb{B}$ be a subset of $\binom{V}{k}$. We assume $1 \le t \le k$. Then a pair $(V, \mathbb{B})$ is called a combinatorial t-design (t -(v, k, λ) design), if the following condition is satisfied.
For each subset T in $\binom{V}{t}$, the number $|\{B \in \mathbb{B} \mid T \subset B\}|$ is constant λ, i.e., does not depend on the choice of T.

As you may know, 2-$(q^2 + q + 1, q + 1, 1)$ is a projective plane of order q, and 5-$(24, 8, 1)$ is the Witt design related to the Mathieu group of degree 24.

The following fact is well known (Delsarte, 1973). Combinatorial t-(v, k, λ) design is equivalent to a t-design in the Johnson association scheme. Note that for the Johson association scheme $J(v, k)$, $X = \binom{V}{k}$, and let Y be a t-design on $J(v, k)$. Then the pair (V, Y) is a combinatorial t-design, and vice versa.

The concept of t-design in the Hamming association scheme $H(d,q)$ is equivalent to that of so-called orthogonal arrays. Let $X = F \times \cdots \times F$ be the vertex set of $H(d,q)$. A subset Y of X is called an orthogonal array if it satisfies the following condition.

If we fix $z = (z_1, z_2, \ldots, z_t) \in F^t$ and a set

$$L = \{\ell_1, \ell_2, \ldots, \ell_t\}$$

of $\{1, 2, \ldots, d\}$ which consist of t distinct integers, then the cardinality of the set $\{y = (y_1, y_2, ..., y_d) \in Y \mid y_{\ell_i} = z_i \ (i = 1, 2, ..., t)\}$ equals a constant λ which is independent of the choices of z and L.

The constant λ is called the index of the orthogonal array. If t is the largest integer which satisfied this condition, t is called the strength of the orthogonal array Y.

Usually, an orthogonal array Y is displayed as a $|Y| \times d$-mtrix where each element of Y is a row vector. We can show that Y is a t-design in the Q-polynomial association scheme $H(d,q)$ if and only if the strength of the orthogonal array Y is at least t.

For a subset Y of X in a Q-polynomial association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$, the characteristic matrix H_k is defined as follows. Let $\phi_{k,1}, \phi_{k,2,}, \ldots, \phi_{k,m_k}$ be the orthonormal basis of the space spanned by column vectors of E_k. Then the characteristic matrix H_k is the matrix of size $|Y| \times m_k$, whose (y,i)-entry is defined by $\phi_{k,i}(y)$ where $y \in Y$ and $1 \le i \le m_k$. Then we have:

(1) $||{}^t H_k H_0||^2 = 0$ for $1 \le k \le t$, if and only if Y is a t-design.

(2) For $x, y \in Y$, let $(x,y) \in R_j$. Then

$$\sum_{i=0}^{m_k} \phi_{k,i}(x)\phi_{k,i}(y) = Q_k(j) \text{ (Addition Theorem) } .$$

(Note that here $Q_k(i)$ is not the Gegenbauer polynomal, but the entry of the second eigenmatrix Q of the association scheme $\mathfrak{X}$.)

So, by a similar argument as we proved the Fisher type lower bound (for spherical $2e$-designs), we get the Fisher type lower bound for the size of a t-design in a Q-polynomial association scheme, as explained in the next page.

Fisher type lower bound of t-designs in Q-polynomial association schemes.

Theorem. (Delsarte, 1973).
Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme. Let $Y \subset X$ be a $2e$-design in $\mathfrak{X}$. Then we have

$$|X| \geq m_0 + m_1 + \cdots + m_e,$$

where $m_i = \mathrm{rank}(E_i)$.

We call X is a tight $2e$-design in the Q-polynomial association scheme, if the equality holds in the above inequality.

Theorem. (Delsarte, 1973).
Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme. Let $Y \subset X$ be a tight $2e$-design in $\mathfrak{X}$. Then the zeros of the polynomial $v_0^*(x) + v_1^*(x) + \cdots + v_e^*(x)$ must be in the set $\{\theta_1^*, \theta_2^*, \cdots, \theta_e^*\}$ where $\theta_i^* = Q_1(i)$, the entries of the i-th column of the second eigenmatrix Q of the association scheme $\mathfrak{X}$.

In the next section, we state what are the current status of the classification problems of tight t-designs in $J(v, d)$ and $H(d, q)$. We will also mention the current status for other known P-and Q-polynomial association schemes.

Classification of tight combinatorial t-designs.

It is known (first remarked by Delsarte) that t-designs in the Johnson association scheme $J(v,k)$ are the same as the combinatorial t-(v,k,λ) design for some λ. Since $m_i = \binom{v}{i} - \binom{v}{i-1}$ for $J(v,k)$, a tight $t(=2e)$-design has size $\binom{v}{e}$. The following is a brief history of the classification problem of tight combinatorial $2e$-designs. For simplicity we consider the case of t is even. The case for t odd is a kind of follows if the case t(=even) is solved.

(1) $e = 1$, there are many examples (i.e., symmetric 2-designs) and the classification seems to be hopeless.
(2) $e = 2$, there are exactly two such designs, i.e., the 4-$(23, 7, 1)$ designs and the 4-$(23, 16, 52)$ design (Enomoto-Ito-Noda, 1975). (For a new simplified proof by Noda, see the book of BBIT, De Gruyter, 2021.)
(3) $e = 3$, there are no tight 6-designs (Peterson, 1977).
(4) $e = 4$, there are no tight 8-designs (Z. Xiang, 2018).
(5) For each fixed $e \geq 5$, there are only finitely many tight $2e$-designs (Bannai, 1977).
(6) For $5 \leq e \leq 9$, there are no tight $2e$-designs (Dukes and Short-Gershman, 2013).
(7) Main Theorem (Ziqing Xiang, arXiv:2312.14778). There are no tight $2e$-designs for $e \geq 10$. (This completely solved the classification problem of tight $2e$-designs for all $e \geq 2$.)

The classification of tight t-designs in $H(n,q)$.

Tight $2e$-design in $H(n,q)$ is an orthogonal array of strength $2e$. The Fisher type lower bound $|X| \geq m_0 + m_1 + \cdots + m_e = \sum_{i=0}^{e} \binom{n}{i}(q-1)^i$ was known earlier by C. R. Rao (1947). The classification of tight $2e$-designs for $e \geq 3$ was finally obtained by Y. Hong (1986) except possibly $q = 2$. (Interestingly enough, the classification of tight $2e$-designs in $H(n,2)$ (only for $q = 2$) is still open. The classification of tight 4-designs was studied by R. Noda (1979), and the complete classification was finally obtained by Gavrilyuk-Suda-Viladi (2020). (Similar remark for odd t, as in the case of Johnson association scheme. There is work by Noda (1986) for $t = 5$ and $t = 3$.)

For known families of P-and Q-polynomial association scheme, L. Chihara (1987) proved the general non-existence results of tight t-designs except for very small t. On the other hand, it seems that the general non-existence result of tight t-designs (for large t) for unknown P-and Q-polynomial association scheme is still open, I think.

Leonard's Theorem for P-and Q-polynomial association schemes.

The classification problem of P-and Q-polynomial association schemes was one of the most important target of the theory of association schemes, as well as algebraic combinatorics. This means that there are two sets of (discrete) orthogonal polynomials $\{v_i(x)\}$ and $\{v_i^*(x)\}$ and they are related by the relation

$$\frac{v_i(\theta_j)}{k_i} = \frac{v_j^*(\theta_i^*)}{m_j}.$$

Then what are these sets of orthogonal polynomials satisfying these restrictions, purely at the level of orthogonal polynomials, forgetting about association schemes? Doug Leonard (1982, 1984) answered to this question, by finding that they are described by small number of free parameters and they must be described by Askey-Wilson orthogonal polynomials, or their special cases or their limiting cases. (Askey- Wilson polynomials were extremely important classes of orthogonal polynomials described by using basic hyper-geometric series (with some base q) and in addition these orthogonal polynomials were discovered just before Leonard obtained his theorem). Leonard's theorem was a big breakthrough, and this made a big bridge between algebraic combinatorics and orthogonal polynomials. Then, Bannai-Ito (1984) Algebraic Combinatorics I, made the detailed description of these families of orthogonal polynomials, dividing essentially into the three cases, $q \neq \pm 1, q = 1$ and $q = -1$. Then P. Terwilliger worked out more, and published the following papers first in 2001 and again in 2021.

P. Terwilliger : Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear Algebra Appl. (2001).

P. Terwilliger : Notes on the Leonard system classification. Graphs Comb. (2021).

Askey-Wilson orthogonal polynomials and Askey scheme (tableau).

The following paper studies P-and Q-polynomial association schemes (Leonard pairs) from the view point of Askey scheme.

P. Terwilliger: An algebraic approach to the Askey scheme of orthogonal polynomials, in "Orthogonal polynomials and special functions. Computation and applications", Lecture Notes in Mathematics 1883, 255-330 (2006).

This paper gives a description of the intimate connection between certain Leonard pairs and a special class of orthogonal polynomials (the 'terminating' branch of the Askey scheme: q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, affine q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bennai/Ito and orphan polynomials).

The Askey scheme is a way of organizing orthogonal polynomials of hypergeometric or basic hypergeometric type into a hierarchy. At the top of its scheme lies the q-Racah polynomials (namely, Askey-Wilson polynomials). So the detailed descriptions of Leonard's theorem (classification of Leonard pairs) occupied a very important role in Askey scheme. Anyway, it would be very interesting to study the classification problem of P-and Q-polynomial association schemes from the viewpoint of Askey scheme of orthogonal polynomials.

(To be continued)