G2C2 Lecture No. 1. Spherical t-designs.

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August 12, 2024

Basic References

[1] Delsarte-Goethals-Seidel: Spherical codes and designs, Geom. Dedicata (1977).

[2] Bannai-Bannai-Ito-Tanaka: Algebraic Combinatorics, Chapter 5, De Gruyter (2021).

We will give more references later.

Spherical codes and designs, in particular spherical t-designs

$S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$,
$X \subset S^{n-1}$, $0 < |X| < \infty$.

The minimum distance $d(X)$ of X is defined by

$$d(X) = \mathrm{Min}\,\{d(x, y) \mid x, y \in X, x \neq y\}.$$

Here, $d(x, y) = \|x - y\|$ (Euclidean distance in $\mathbb{R}^n$). We also consider the inner product (x, y), and the angle $\theta(x, y)$ (= geodesic distance).

Note that they are related by $\cos(\theta(x, y)) = (x, y)$ and $(x, y) = 1 - \frac{1}{2}d(x, y)^2$

To study finite set X of S^{n-1}, there are (at least) two different viewpoints: (i) Coding theoretical viewpoint and (ii) Design theoretical viewpoint. The coding theoretical view is to find a finite subset that is scattered as much as possible, i.e., points are separated as much as possible. The design theoretical view is to find a finite subset that approximates the whole space as much as possible.

(i) Coding theoretical viewpoint

(a) Tammes Problem.

$|X|$ is given. Make the minimum distance $d(X)$ of X as large as possible.

What will happen, if $|X| = 1, 2, 3, 4, 5, 6, \ldots\ldots$?

The answer is known for $|X| \leq 14$ and $|X| = 24$.
(See Ericson-Zinoviev: Codes on Euclidean Spheres (2001).)

The following is a quote from: The Tammes problem for $N = 14$, by Oleg R. Musin, Alexey S. Tarasov, Exp. Math. (2015).

The Tammes problem is to find the arrangement of N points on a unit sphere which maximizes the minimum distance between any two points. This problem is presently solved for several values of N, namely for $N = 3, 4, 6, 12$ by L. Fejes Toth (1943); for $N = 5, 7, 8, 9$ by Schutte and van der Waerden (1951); for $N = 10, 11$ by Danzer (1963) and for $N = 24$ by Robinson (1961). Recently, we solved the Tammes problem for $N = 13$. The optimal configuration of 14 points was conjectured more than 60 years ago. In the paper, we give a solution of this long-standing open problem in geometry. Our computer-assisted proof relies on an enumeration of the irreducible contact graphs.

(b) Kissing Number Problem.
The minimum distance $d(X)$ of X is given. Make $|X|$ as large as possible.

If the minimum distance $d(X)$ of X is 1 in Euclidean distance (or equivalently $\pi/3(=60^\circ)$ in the geodesic distance, or the inner product is $1/2$), then we have the famous Kissing number problem in $\mathbb{R}^n$.

$k(2) = 6$ (obvious),
$k(3) = 12$ (There is a long history for this problem starting Newton-Gregory dispute in 1694. Shütte-van der Waerden, 1953),
$k(4) = 24$ (Musin, 2008),
$k(8) = 240$ (Odlyzko-Sloane, Levenshtein, 1979),
$k(24) = 196560$ (Odlyzko-Sloane, Levenshtein, 1979).
($k(n)$ for other values of n are open !)

There are some other problems of coding theoretical nature.

(c) s-distance set problem on S^{n-1}.

$X \subset S^{n-1}$ is called an s-distance set on S^{n-1},
if $|\{d(x,y) \mid x,y \in X,\ x \neq y\}| = s$
(or some people allow this number to be at most s, instead of exactly s.)

Problem. If X is a 1-distance set in S^{n-1}, then $|X| \leq n+1$.
Moreover, if $|X| = n+1$, then X must be a regular simplex. (Exercise.)

Results. (i) It is known that if X is an s-distance set on S^{n-1}, then

$$|X| \leq \binom{n-1+s}{s} + \binom{n-1+s-1}{s-1}.$$

(ii) It is an interesting problem to classify s-distance set X on S^{n-1} with
$|X| = \binom{n-1+s}{s} + \binom{n-1+s-1}{s-1}$.

Examples are known for $(s,n) = (2,6)$ and $(s,n) = (2,22)$.
It is known that for $s \geq 3$ there is no s-distance set X on S^{n-1} with
$|X| = \binom{n-1+s}{s} + \binom{n-1+s-1}{s-1}$.
(The classification problem is still open for $s = 2$.)
(It is known that n must be (an odd integer)$^2 - 3$. The case $n = 118$ is the smallest open case. Namely, non-existence for $n = 46, 78$.)

Related Problems.

(1) We can consider s-distance sets in $\mathbb{R}^n$ instead of S^{n-1}. (Or, for any metric space.)
(Question: Can you find 2-distance sets X in $\mathbb{R}^3$ with $|X| = 6$?
Can you find how many different such examples are there ?)
It is known that if X is an s-distance set in $\mathbb{R}^n$, then $|X| \leq \binom{n+s}{s}$.

(1-a) Can you prove for $s = 1$. Can you classify those with $|X| = n + 1 (= \binom{n+1}{1})$.

(1-b) One example for $s = 2$ with $|X| = \binom{n+2}{2}$ is known for $n = 8$ (Lisoněk, 1997). No other example with $|X| = \binom{n+s}{s}$ is known for any $s \geq 2$ and $n \geq 2$. (Whether there is any other example is a famous open problem.)

It seems that the classifications of maximum 2-distance sets in $\mathbb{R}^n$ have been obtained recently by Chin-Yen Lee, Yu-Hang Sheng, Meng-Tsung Tsai, and Wei-Hsuan Yu, for $n \leq 14$, generalizing the work of Lesoněk. (The paper is in preparation.)

Further related problems. (Homework problems)

We call a subset X in $\mathbb{R}^n$ an s inner product set in $\mathbb{R}^n$, if $|\{x \cdot y \mid x, y \in X, x \neq y\}| = s$.

(1-i) Can you prove that $|X| \leq n+1$, if X is a 1-inner product set ?

(1-ii) Can you find some examples of 1-inner product sets in $\mathbb{R}^n$? For $n = 2$? For $n = 3$? For $n = 4$?

(1-iii) Suppose that X is on two concentric spheres with the centers at the origin. Can you characterize such 1-inner product sets in $\mathbb{R}^n$ with $|X| = n+1$?

More generally, can you characterize general 1-inner product sets in $\mathbb{R}^n$ with $|X| = n+1$? (May be difficult !)

It is known that if X is an s-inner product set in $\mathbb{R}^n$, then $|X| \leq \binom{n+s}{s}$. (Deza-Frankl, 1985, Proc. AMS. See also Nozaki, 2011 Combinatorica.)

No example of an s-inner product set in $\mathbb{R}^n$, with $|X| = \binom{n+s}{s}$ is known for any pair (s, n) with $s \geq 2$ and $n \geq 2$.
(I think this is an open problem, but I believe no one has tried seriously to find examples with $|X| = \binom{n+s}{s}$. Please try to find some examples !)

Possible Research Problem! We may be able to consider *s*-distance sets in more general metric spaces.

Fedor Petrov, Cosmin Pohoata: A remark on sets with few distances in $\mathbb{R}^d$. Proc. Am. Math. Soc. 149 (2021).

Gábor Hegedüs, Lajos Rónyai: An upper bound for the size of *s*- distance sets in real algebraic sets. Electron. J. Comb. 28 (2021).

Some other problems of coding theoretical flavor.

(d) Coulomb-Thomson Problem.
Let N be a natural number. Among all N-element subset $X = \{x_1, x_2, ..., x_N\}$ of S^{n-1}, find subsets that minimize the following value

$$\sum_{1 \le i < j \le N} \frac{1}{\|x_i - x_j\|}.$$

Also, determine this minimum value as well as the structure of such sets.
It is possible to consider a similar problem for many other energy functions.

(e) The covering radius Problem.
Let N be a natural number. Among all N-element subset $X = \{x_1, x_2, \ldots, x_N\}$ of S^{n-1}, find subsets that minimize the following value:

$$\mathrm{Max}_{x \in S^{n-1}} \{\mathrm{Min}_{1 \le i \le N}(d(x, x_i))\}$$

This minimum value is called the covering radius of the set X. Also, determine this minimum value as well as the structure of such sets.

Design theoretical viewpoint

(We want to approximate the whole space by a smaller finite subset.)

Spherical t-designs(Delsarte-Goethals-Seidel, 1977).

A finite subset $X \subset S^{n-1}$ is called a spherical t-design on S^{n-1}, if one of the following equivalent conditions is satisfied, where t is a positive integer.

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x),$$

for ${}^{\forall} f(x) = f(x_1, x_2, \ldots, x_n)$, polynomials of degree $\leq t$,

$$\Longleftrightarrow \sum_{x\in X} f(x) = 0 \text{ for } {}^{\forall} f(x) \in \mathrm{Harm}_i(\mathbb{R}^n),\ 1 \leq i \leq t,$$

$$\Longleftrightarrow \sum_{(x,y)\in X\times X} Q_i(x\cdot y) = 0,\ 1 \leq i \leq t,$$

$$(Q_i(x) = \text{Gegenbauer polynomial})$$

$\Longleftrightarrow$ any moment of X of degree $\leq t$ is invariant under orthogonal transformations,

($\Longleftrightarrow$ many other equivalent conditions.)

(1) Examples of t-designs on S^{n-1}

For $n = 2$, the $t + 1$ vertices of a regular $(t + 1)$-gon inscribed on $S^1 (\subset \mathbb{R}^2)$ form a t-design. (So, we mostly consider the cases $n \geq 3$ in what follows.)

For $n = 3$, the set of vertices of a regular polyhedron X inscribed in S^2 are, spherical $2, 3, 3, 5, 5$-designs for regular simplex (4 vertices), regular octahedron (6 vertices), cube (8 vertices), regular icosahedron (12 vertices), and regular dodecahedron (20 vertices), respectively.

Challenging Problems. Can you find some spherical t-designs for large t in S^2 ?

Many examples are obtained either as

(a) an orbit of a finite group $G \subset O(n)$, or

(b) a shell of a lattice $L \subset \mathbb{R}^n$,
i.e., $X = \{\frac{1}{\sqrt{m}} x \mid x \in L,\ x \cdot x = m\}$ for a fixed m.

However, those known examples are always

$t \leq 19$ for (a) (for $n \geq 3$)

$t \leq 11$ for (b) (for any n).

It is open whether t is always bounded by an absolute constant independent of n in each of the cases (a) and (b) !

Some easy looking problems.

- **Is there any spherical 6-design on a shell of a lattice L in $\mathbb{R}^2$?**

- **Is there any spherical 4-design on a shell of a lattice L in $\mathbb{R}^3$?**

- **Is there any spherical 4-design on S^1 whose coordinates are all rational numbers? (Also the same question for bigger t?)**

(2) Existence of spherical t-design on S^{n-1}

- **There exist t-designs on S^{n-1} for $^\forall n$ and $^\forall t$!**
  (Seymour-Zaslavsky, Advances in Math., 1984)

- There are many proofs known, but they are mostly existence theorems, and good explicit constructions are not known.

- The best existence result is due to Bondarenko-Radchenko-Viazovska (Annals of Math.,2013) that shows the existence of spherical t-designs on S^{n-1} with the sizes asymptotically the same order as the best possible bound, if n is fixed and $t \to \infty$. Namely, they showed that there is a constant c_{n-1} for each n such that for each integer $N \geq c_{n-1}t^{n-1}$, there exists a spherical t-design X of size $|X| = N$. (How about the case when t is fixed and $n \to \infty$? I think this is still open.)

- Most of known existence proofs use the continuous property of real numbers. A new existence proof was obtained by Zhen Cui, Jiacheng Xia, and Ziqing Xiang: "Rational designs" (Advances in Math., 2019) that uses analytic number theory: Hilbert-Kampe problem.

(3) <u>Explicit construction of spherical t-designs on S^{n-1} ($n \geq 3$)</u>

- Some are known (Kuperberg, 2005, for $n = 3$). See also, for $n = 3$, $|X| = (t + 1)^2$ with $t \leq 100$ (Chen-Frommer-Lang, 2011)

- General Case: Ziqing Xiang, Explicit spherical designs, (Algebraic Combinatorics, 2022)

- See also Bannai-Nakata-Okuda-Zhao: Explicit construction of exact unitary designs, Advances in Math. 2022.

(4) Lower bounds for $|X|$ (Fisher type lower bound)

$$|X| \geq \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}, \text{ if } t = 2e,$$

$$|X| \geq 2\binom{n-1+e}{e}, \qquad \text{if } t = 2e+1,$$

" = " holds $\Longleftrightarrow$ X is called a tight spherical t-design.

(5) Let X be a t-design and s-distance set, i.e.,
$s = |A(X)|$, where $A(X) = \{x \cdot y \mid x, y \in X, x \neq y\}$. Then
(i) $t \leq 2s$.
(ii) $t = 2s \Longleftrightarrow X$ is a tight $2s$-design.
(iii) $t = 2s-1$ and X is antipodal $\Longleftrightarrow X$ is a tight $(2s-1)$-design.
(iv) $t \geq 2s-2 \Longrightarrow X$ has the structure of a Q-polynomial association scheme.
(v) $t \geq 2s-3$ and X is antipodal $\Longrightarrow X$ has the structure of a Q-polynomial association scheme.

Classification of tight t-design on S^{n-1}

$n = 2 \implies X$ is a regular $(t+1)$-gon
(So we assume $n \geq 3$ in what follows)

- We get $t \in \{1, 2, 3, 4, 5, 7, 11\}$ (Bannai-Damerell, 1979,1980).

Tight t-designs on S^n are classified for all t, except $t = 4, 5, 7$.
Some further non-existence results for $t = 4, 5, 7$ are known.
(Bannai-Munemasa-Venkov (2004), Nebe-Venkov (2013).)
But the problem is still open for $t = 4, 5, 7$.

$t = 1 \implies X$ is an antipodal pair $(|X| = 2)$
$t = 2 \implies X$ is a regular simplex $(|X| = n + 1)$
$t = 3 \implies X$ is a cross-polytope $(|X| = 2n)$
$t = 4 \implies n = (\text{an odd integer})^2 - 3$, and only two examples with $(n, |X|) = (6, 27),\ (22, 275)$ are known

$t = 5 \Longrightarrow n = 3$ or $n = (\text{an odd integer})^2 - 2$,
and only three such examples with $(n, |X|) = (3, 12), (7, 56)$, $(23, 552)$ are known.

$t = 7 \Longrightarrow n = 3(\text{an integer})^2 - 4$, and
only two such examples with $(n, |X|) = (8, 240)$, $(23, 4600)$ are known.

$t = 11 \Longrightarrow (n, |X|) = (24, 196560)$, and X is unique (Bannai-Sloane (1981)).

To be Continued.

Thank you !