Graphs and Groups, Complexity and Convexity (G2C2-2024)

(Summer School, August 11-25, 2024, Hebei Normal University, Shijiazhuang, China)

Title: An overview on design theory - explicit constructions of designs and the classification problems of tight designs.

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Description: Delsarte theory of codes and designs has been developed over association schemes (Delsarte, 1973) and over spheres (Delsarte-Goethals-Seidel, 1977). There are close similarities between these two situations. We first explain these theories, emphasizing the Fisher type inequalities and the classification problems of tight t-designs. Then we study various generalizations of these concepts: Euclidean t-designs and relative t-designs, and the classification problem of tight t-designs. We want to give an overview on how much these design concepts are generalized. We will also discuss the explicit construction problem of t-designs (for unitary t-designs and spherical t-designs). Finally, we hope to aim towards the classification problems of finite Gelfand pairs and commutative association schemes.

Outline of the course:

Lecture 1. Spherical t-designs.
We start with the basics of spherical t-designs and spherical codes following Delsarte-Goethals-Seidel (1977), including spherical harmonics and Gegenbauer polynomials, etc.

Lecture 2. t-designs on Q-polynomial association schemes.
We explain the theory of codes and designs on association schemes following Delsarte (1973), in particular t-designs on Q-polynomial association schemes. We will also discuss P-and Q-polynomial association schemes, Leonard pairs, Askey-Wilson orthogonal polynomials, etc.

Lecture 3. The classification problems of tight spherical t-designs and tight combinatorial t-designs.
We discuss the concept of tight t-designs, and the classification problems of tight spherical t-designs and tight combinatorial t-designs (and tight t-designs on Q-polynomial association schemes).

Lecture 4. Euclidean t-designs, and the classification problems of tight Euclidean t-designs.
We discuss the concept of Euclidean t-designs and the classification problems of certain tight Euclidean t-designs.

Lecture 5. Relative t-designs in association schemes.
We discuss the concept of relative t-design in association schemes due to Delsarte (1977), and how this concept is similar to that of Euclidean t-design.

Lecture 6. Design theory from the viewpoint of algebraic combinatorics.
We discuss various generalizations of the concept of t-design, following Bannai-Bannai-Tanaka-Zhu (2017).

Lecture 7. The explicit constructions of unitary t-designs.
We give the definition of unitary t-designs, and give the explicit constructions of them (following Bannai-Bannai-Nakata-Zhao). As a by-product, we can also get the explicit constructions of spherical t-designs.

Lecture 8. An overview on finite Gelfand pairs (and commutative association schemes).
We explain the current situation toward the classification problems of finite Gelfand pairs (and commutative association schemes). We also discuss multivariable P-polynomial and Q-polynomial association schemes (following Bernard-Crampe-d'Andecy-Vinet-Zaimi and Bannai-Kurihara-Zhao-Zhu, see also Ceccherini-Silberstein-Scarabotti-Tolli).

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